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Schur's Algorithm, Orthogonal Polynomials, and Convergence of Wall's Continued Fractions in $L^2(\mathbb{T})$

Sergei Khrushchev

Dolgoozernaya ul. 6, Block 1, Apt. 116, 197373 St. Petersburg, Russia

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A function f in the unit ball $\mathscr{B}$ of the Hardy algebra H^∞ on the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ is a non-exposed point of $\mathscr{B}$ ($|f|<1$ a.e. on $\mathbb{T}=\{\zeta\in\mathbb{C}:|\zeta|=1\}$) iff $\lim_n \int_{\mathbb{T}} |f_n|^2\,dm=0$, where m is the Lebesgue measure on $\mathbb{T}$ and $(f_n)_{n\geqslant 0}$ are the Schur functions of f. This result easily implies Rakhmanov's well-known theorem which states that $\lim_n a_n=0$ if $\sigma'>0$ a.e. on $\mathbb{T}$, $(a_n)_{n\geqslant 0}$ being the parameters of the orthogonal polynomials $(\varphi_n)_{n\geqslant 0}$ in $L^2(d\sigma)$. We prove that $f_n b_n$ is the Schur function of the probability measure $|\varphi_n|^2\,d\sigma$, which leads to an important formula relating $|\varphi_n|^2\,\sigma'$ to f_n and $b_n=\varphi_n/\varphi_n^*$. A probability measure σ is called a Rakhmanov measure if $(*)-\lim_n |\varphi_n|^2\,d\sigma=dm$. We show that a probability measure σ with parameters $(a_n)_{n\geqslant 0}$ is a Rakhmanov measure iff the a_n's satisfy the Máté–Nevai condition $\lim_n a_n\,a_{n+\kappa}=0$ for every $\kappa=1,2,\ldots$. Next, we prove that even approximants A_n/B_n of the Wall continued fraction for f converge in $L^2(\mathbb{T})$ iff either f is an inner function or $\lim_n a_n=0$. This implies that measures satisfying $\lim_n a_n a_{n+\kappa}=0$, $\kappa=1,2,\ldots$, and $\overline{\lim}_n |a_n|>0$ are all singular. © 2001 Academic Press

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1. INTRODUCTION

In his famous memoirs [8, Sects. 356–382, Chap. 18] L. Euler presented the first systematic study of continued fractions, which he prefaced with his strong belief that someday applications of continued fractions would be widespread in the analysis of infinities. The ensuring development of mathematics confirmed Euler's prediction. A brief history of the subject can be found in [23, Sect. 1.1]. However, already in 1938 G. Szegő wrote in the Preface to his well-known book [51]: "Despite the close relationship between continued fractions and the problem of moments, and notwithstanding recent important advances in this latter subject, continued fractions have been gradually abandoned as a starting point for the theory of orthogonal polynomials." Nowadays this tendency has only increased. Continued fractions are considered a cumbersome tool deserving to be expelled from consideration. Continued fractions could also be expelled from the present paper. However, it does not look reasonable to artificially exclude a fascinating, object integrating such different at first sight areas as Schur's algorithm, orthogonal polynomials, and the Euclidean algorithm.

One of the most beautiful results presented in [8, Sect. 371] is the formula

$$\sum_{\kappa=0}^{n} \gamma_\kappa z^\kappa = \frac{\gamma_0}{1}\,\frac{(\gamma_1/\gamma_0)\,z}{-1+(\gamma_1/\gamma_0)\,z-}\,\cdots\,\frac{(\gamma_n/\gamma_{n-1})\,z}{-1+(\gamma_n/\gamma_{n-1})\,z}, \tag{1.1}$$

representing the partial sums of a Taylor series as the approximants of a continued fraction. Euler's formula (1.1) can easily be obtained from Euler's recurrence formulae for the numerators P_n and the denominators Q_n of a continued fraction $q_0 + \mathrm{K}_{n=1}^{\infty}(p_n/q_n)$,

$$P_n = q_n P_{n-1} + p_n P_{n-2}, \qquad Q_n = q_n Q_{n-1} + p_n Q_{n-2}, \tag{1.2}$$

$n = 1, 2, ...,$ where $P_{-1} = 1$, $P_0 = q_0$, $Q_{-1} = 0$, $Q_0 = 1$. Indeed, assuming we are given a Taylor polynomial (1.1), we may assume that $Q_0 = \cdots = Q_n = 1$. Then the required continued fraction must satisfy $P_\kappa = \sum_{j=0}^{\kappa} \gamma_j z^j$, $\kappa = 0, 1, ..., n$. Resolving the system of linear equations (1.2), we obtain that

$$q_0 = \gamma_0, \qquad p_1 = \gamma_1 z, \qquad q_1 = 1$$
$$p_\kappa = -(\gamma_\kappa/\gamma_{\kappa-1})\,z, \qquad q_\kappa = 1 + (\gamma_\kappa/\gamma_{\kappa-1})\,z, \qquad \kappa = 2, 3, ..., n.$$

Applying an elementary identity

$$\gamma_0 + \frac{\gamma_1 z}{1+w} = \frac{\gamma_0}{1}\,\frac{(\gamma_1/\gamma_0)\,z}{-1+(\gamma_1/\gamma_0)\,z+w}$$

to the continued fraction obtained, we complete the proof of (1.1).

In spite of its simplicity Euler's formula has many important applications starting from Brouncker's formula,

$$\frac{\pi}{4}=\frac{1}{1+}\frac{1}{2+}\frac{3^2}{2+}\frac{5^2}{2+}\cdots\frac{(2n-1)^2}{2+}\cdots,$$

which was derived by Euler [8, Sect. 369] from the Taylor expansion of arctg z at $z=0$, to the fundamental inequalities in the convergence theory of continued fractions [23, Sect. 4.4.5, 26].

It is interesting that Schur's classical algorithm can be put in the form of a continued fraction which is very similar to Euler's continued fraction (1.1).

Let $\mathscr{B}$ be the set of all functions f holomorphic on the unit disc $\mathbb{D} \overset{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\}$ and satisfying

$$\|f\|_\infty \overset{\text{def}}{=} \sup\{|f(z)|: z \in \mathbb{D}\} \leqslant 1.$$

Clearly, $\mathscr{B}$ is the unit ball of the Hardy algebra H^∞. See [12] for the basic facts on H^∞. Recall [12, Chap. IV, Example 21] that for every f in $\mathscr{B}$, which is not a finite Blaschke product, Schur's algorithm determines an infinite sequence $(\gamma_n)_{n \geqslant 0}$, $\gamma_n \in \mathbb{D}$, as follows

$$f(z) \overset{\text{def}}{=} f_0(z)=\frac{zf_1(z)+\gamma_0}{1+\bar{\gamma}_0 z f_1(z)}; \dots ; f_n(z)=\frac{zf_{n+1}(z)+\gamma_n}{1+\bar{\gamma}_n z f_{n+1}(z)}; \dots. \tag{1.3}$$

In case

$$f=\prod_{\kappa=1}^{n}\frac{|\lambda_\kappa|}{\lambda_\kappa}\cdot\frac{\lambda_\kappa - z}{1-\bar{\lambda}_\kappa z}$$

is a finite Blaschke product with n zeros $\lambda_1, \dots, \lambda_n$ in $\mathbb{D}$ we have $|\gamma_n|=1$ in (1.3) and Schur's algorithm interrupts at the nth step by Schwarz's lemma [12].

The numbers $\gamma_n = f_n(0)$, $n=0, 1, \dots$, are called *the Schur parameters* of f and the functions f_n are called *the Schur functions.*

By (1.3) f_n is a superposition of f_{n+1} and of the Möbius transform

$$\tau_n(w)=\frac{zw+\gamma_n}{1+\bar{\gamma}_n zw}=\gamma_n+\frac{(1-|\gamma_n|^2)\,z}{\bar{\gamma}_n z+1/w},$$

which for every z, $z \in \mathbb{D}$, maps the closed disc $\{w: |w| \leqslant 1\}$ onto a closed disc in $\mathbb{D}$. Iterating we obtain that

$$f(z)=\tau_0 \circ \tau_1 \circ \cdots \circ \tau_n(f_{n+1}), \tag{1.4}$$

which obviously can be put in the form of a continued fraction

$$f(z)=\gamma_0+\frac{(1-|\gamma_0|^2)\,z}{\bar{\gamma}_0 z}\;\frac{1}{+\gamma_1+}\;\frac{(1-|\gamma_1|^2)\,z}{\bar{\gamma}_1 z}\;\frac{\cdots}{+}\;\frac{1}{+\gamma_n+}\;\frac{(1-|\gamma_n|^2)\,z}{\bar{\gamma}_n z}\;\frac{\cdots}{+}. \tag{1.5}$$

Such a representation of Schur's algorithm was obtained by Wall [54] (received by the editors May 26, 1943). Wall also proved that the approximants A_n/B_n of order $2n$ for (1.5) converge to f *uniformly on compact subsets* of $\mathbb{D}$; notationally,

$$\frac{A_n}{B_n} \rightrightarrows f \tag{1.6}$$

Notice that the quotient A_n/B_n is obtained if we interrupt (1.5) at the term $1/\gamma_n$ and then make all arithmetic operations without cancellations. Hence A_n and B_n are polynomials in z of degree n. In what follows A_n, B_n are called *Wall polynomials.*

At approximately the same time Geronimus [13] (received by the editors March 18, 1943) obtained another decomposition of f into a continued fraction,

$$f(z)=\frac{\gamma_0}{1\,-}\;\frac{(1-|\gamma_0|^2)(\gamma_1/\gamma_0)\,z}{1+(\gamma_1/\gamma_0)\,z}\;\frac{}{-}\;\frac{(1-|\gamma_1|^2)(\gamma_2/\gamma_1)\,z}{1+(\gamma_2/\gamma_1)\,z}\;\frac{\cdots}{-}\;\frac{}{-}\;\frac{(1-|\gamma_{n-1}|^2)(\gamma_n/\gamma_{n-1})\,z}{1+(\gamma_n/\gamma_{n-1})\,z}\;\frac{\cdots}{-}, \tag{1.7}$$

which in fact coincides with the even part of (1.5). In other words the approximant of order n for (1.7) is exactly A_n/B_n. Geronimus [13] also proved (1.6).

Returning to Euler's continued fraction (1.1), one can observe a remarkable similarity between (1.7) and (1.1). Since any continued fraction satisfies [23, Theorem 2.1, Sect. 2.1]

$$\frac{P_n}{Q_n}-\frac{P_{n-1}}{Q_{n-1}}=\frac{(-1)^{n-1}\,P_1\cdots P_n}{Q_n Q_{n-1}}, \tag{1.8}$$

we arrive to the conclusion that f and A_n/B_n have the same Taylor polynomials of order n centered at $z=0$. The above example supports the expectations that the continued fractions (1.5) and (1.7) also may have many important applications. These expectations were already justified in [13]. Using (1.7), Geronimus [13] obtained an important formula relating the Schur parameters of f, $f \in \mathcal{B}$, with the parameters of orthogonal polynomials. To state Geronimus' theorem we need some preliminaries.

Let $\mathbb{T} = \{z: |z| = 1\}$ be the unit circle. Given a probability measure σ on $\mathbb{T}$ the orthogonal polynomials $(\varphi_n)_{n \geqslant 0}$ in $L^2(d\sigma)$ are obtained as the outcome of the standard Gram–Schmidt orthogonalization algorithm applied to the system of monomials $(z^n)_{n \geqslant 0}$:

$$\varphi_n(z) = k_n z^n + \cdots + \varphi_n(0), \qquad k_n > 0$$
$$\int_{\mathbb{T}} \varphi_n \bar{\varphi}_\kappa \, d\sigma = \begin{cases} 0, & \kappa < n \\ 1, & \kappa = n. \end{cases} \tag{1.9}$$

For a polynomial p, $p \in \mathcal{P}_n$, $\mathcal{P}_n$ being the linear space of all polynomials in z of degree n, we put

$$p^*(z) = z^n \overline{p(1/\bar{z})}. \tag{1.10}$$

It follows from the recurrence formulae [51, Chap. XI, Sect. 11.4, (11.4.6–11.4.7)]

$$k_n \varphi_{n+1} = k_{n+1} z \varphi_n + \varphi_{n+1}(0)\, \varphi_n^*$$
$$k_n \varphi_{n+1}^* = k_{n+1} \varphi_n^* + \overline{\varphi_{n+1}(0)}\, z\varphi_n \tag{1.11}$$

that the orthogonal polynomials $(\varphi_n)_{n \geqslant 0}$ are uniquely determined by the parameters $a_n = -\overline{\varphi_{n+1}(0)}/k_{n+1}$, $n = 0, 1, ...$ We call $(a_n)_{n \geqslant 0}$ the *Geronimus parameters* of σ.

By Herglotz' theorem [49, Theorem 11.12, Theorem 11.19] the Herglotz transform

$$F_\sigma(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\sigma(\zeta) \tag{1.12}$$

is a one-to-one mapping of the set of probability measures on $\mathbb{T}$ onto the set of holomorphic functions F in $\mathbb{D}$ satisfying

$$F(0) = 1, \qquad \operatorname{Re} F(z) > 0, \qquad z \in \mathbb{D}. \tag{1.13}$$

Applying the Möbius transform $(w-1) \cdot (w+1)^{-1}$ to F, we obtain by Schwarz's lemma [12, Chap. I, Lemma 1.1] that the formula

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\sigma(\zeta) = \frac{1 + zf(z)}{1 - zf(z)} \tag{1.14}$$

establishes a one-to-one correspondence between probability measures σ on $\mathbb{T}$ and the elements f of the unit ball $\mathcal{B}$ of the Hardy algebra H^∞. We call the function f in (1.14) *the Schur function* of σ.

THEOREM (Geronimus [13, 14]). *The Geronimus parameters of a probability measure σ on $\mathbb{T}$ coincide with the Schur parameters of the Schur function of σ:*

$$a_n = \gamma_n, \qquad n = 0, 1, \ldots. \tag{1.15}$$

This beautiful result by Geronimus attracted the attention of a number of mathematicians [16, 27, 43] who provided elementary proofs. The original proof, however, used continued fractions.

Since Geronimus' theorem is important for the present paper, we provide its proof in Section 5. Here we demonstrate a typical application of Geronimus' theorem.

THEOREM (Favard [6, 9]). *Any infinite sequence $(a_n)_{n \geqslant 0}$ of points in $\mathbb{D}$ is the sequence of the Geronimus parameters of a probability measure on $\mathbb{T}$.*

Proof. Easy arguments with the normal family $\mathcal{B}$ in $\mathbb{D}$ show that any sequence $(\gamma_n)_{n \geqslant 0}$, $\gamma_n \in \mathbb{D}$, $n = 0, 1, \ldots$ is the sequence of Schur parameters of some function f in $\mathcal{B}$, which uniquely determines a probability measure σ by (1.14). The Geronimus parameters of σ coincide with $(\gamma_n)_{n \geqslant 0}$ by Geronimus' theorem. ∎

Given λ, $|\lambda| = 1$, we denote by σ_λ the probability measure with the Geronimus parameters $(\lambda a_n)_{n \geqslant 0}$. The measure σ_{-1} is of particular interest. We denote by $(\psi_n)_{n \geqslant 0}$ the orthogonal polynomials in $L^2(d\sigma_{-1})$. Substituting (1.7) in (1.14), we obtain by Geronimus' theorem that

$$\begin{aligned} F_\sigma(z) = 1 + \frac{2a_0 z}{1 - a_0 z -} \; \frac{(1 - |a_0|^2)(a_1/a_0)\, z}{1 + (a_1/a_0)\, z \quad -} \; \cdots \\ - \frac{(1 - |a_{n-1}|^2)(a_n/a_{n-1})\, z}{1 + (a_n/a_{n-1})\, z \quad -} \; \cdots. \end{aligned} \tag{1.16}$$

Simple analysis of Euler's recurrence formulae (1.2) for the continued fraction (1.16) and of the recurrence formulae (1.11) for the orthogonal polynomials leads to the conclusion that ψ_n^*/φ_n^* is the approximant of order n for (1.16). This result by Geronimus [13] is an analogue of Tchebyshef's well-known result for orthogonal polynomials on the segment $[-1, 1]$ [52].

The main purpose of the present paper is to apply methods of continued fractions and related ideas to the study of orthogonal polynomials on the unit circle $\mathbb{T}$. Namely, we study the convergence properties of continued fractions (1.7) and prove (Theorem 5, see Section 2 and Section 8) that (1.7) converges in measure (with respect to the normalized Lebesgue measure dm, $\int dm = 1$, on $\mathbb{T}$) if and only if either $\lim_n \gamma_n = 0$ or f is an inner function; i.e., $|f| = 1$ a.e. on $\mathbb{T}$. This theorem is the foundation for "weak" arguments leading to weak asymptotic formulae for orthogonal polynomials. A good illustration is provided by Theorems 7, 8 (see Sects. 2 and 8) which say that

$$\lim_n \int_{\mathbb{T}} \left| \frac{\psi_n^*}{\varphi_n^*} - F_\sigma \right|^p dm = 0 \tag{1.17}$$

for $0 < p < 1$ if and only if either σ is a singular measure on $\mathbb{T}$ or $\lim_n a_n = 0$, $(a_n)_{n \geqslant 0}$ being the Geronimus parameters of σ.

To draw conclusions on "strong" convergence we obtain the following important formula

$$|\varphi_n|^2 \sigma' = \frac{1 - |f_n|^2}{|1 - \zeta b_n f_n|^2} \qquad \text{a.e. on } \mathbb{T}, \tag{1.18}$$

where $(f_n)_{n \geqslant 0}$ are the Schur functions of σ and $b_n = \varphi_n / \varphi_n^*$ (Theorem 2, Sects. 2, 6).

In Sect. 5 we present a new proof of Szegő's classical theorem and obtain by (1.18) its "strong" version (Theorem 2.5 and Sect. 5)

$$\lim_n \int_{\mathbb{T}} \left| \log \frac{1}{|\varphi_n|^2} - \log \sigma' \right| dm = 0 \tag{1.19}$$

for every Szegő measure σ (see Sect. 2 for the definition).

Another application of (1.18) is a new characterization of Erdős measures ($=$ measures on $\mathbb{T}$ with $\sigma' > 0$ a.e. on $\mathbb{T}$) in terms of the corresponding Schur functions. Namely, σ is an Erdős measure if and only if

$$\lim_n \int_{\mathbb{T}} |f_n|^2 \, dm = 0, \tag{1.20}$$

where $(f_n)_{n \geqslant 0}$ is the sequence of Schur's functions of σ (Theorem 1, Sects. 2, 6). This and Geronimus' theorem immediately imply Rakhmanov's well-known theorem [46], which says that the Geronimus parameters of any Erdős measure tend to zero.

In Theorem 3 we extend (1.18) and prove that $b_n f_n$ is the Schur function of the probability measure $|\varphi_n|^2\, d\sigma$ (Sects. 2, 7).

An important rôle in the "weak" part of our approach is played by the so-called Rakhmanov measures (see (2.15)). In Theorem 4 (Sects. 2, 7) we describe Rakhmanov measures in terms of their Geronimus parameters. This description is important for our main result—Theorem 5, since we prove first that for any Rakhmanov measure the continued fraction of its Schur function converges in measure on $\mathbb{T}$.

A special attention is paid to the study of Nevai's class (= measures with $\lim_n a_n = 0$). We show how our approach can be used to derive the most important results for Nevai's class (Sects. 2, 6, 8).

The main technical tools of the present paper are collected in Sects. 4–5. Here we exploit the fact that behind Schur's algorithm and Gram–Schmidt's orthogonalization algorithm stands the algorithm of continued fraction (1.7). See papers [24, 25] by Jones *et al.*, concerning the relationship between the algorithms mentioned.

Notice that convergence result in $L^2(\mathbb{T})$ for Euler's continued fractions is trivial. Indeed, Euler's continued fraction with parameters $(\gamma_n)_{n\geqslant 0}$ converges in $L^2(\mathbb{T})$ if and only if

$$\sum_n |\gamma_n|^2 < +\infty. \tag{1.21}$$

On the other hand, by Boyd's theorem [3] (1.21) is a necessary and sufficient condition for f with Schur parameters $(\gamma_n)_{n\geqslant 0}$ to be either a finite Blaschke product or a nonextreme point of $\mathscr{B}$. The presence of the coefficients $(1-|\gamma_n|^2)$ in (1.7) forces the corresponding continued fraction to converge for a wider class of parameters compared with (1.1).

A generalization of Wall's theorem to more general continued fractions, including the continued fractions corresponding to the polynomials orthogonal with respect to real measures on the unit circle, was considered by Frank [11].

2. THE RESULTS

Recall that a point x in the unit ball of a Banach space X is called an *exposed point* of ball(X) if there is x^* in the conjugate space X^* such that $\|x^*\| = x^*(x) = 1$ but such that $|x^*(y)| < 1$ for all y in ball(X), $y \neq x$. By the Amar–Fisher–Lederer theorem [10, 57] a function f, $f \in H^\infty$, is an exposed point of $\mathscr{B}$ if and only if $\|f\|_\infty = 1$ and $m\{\zeta \in \mathbb{T} : |f(\zeta)| = 1\} > 0$. Our first result describes exposed points of $\mathscr{B}$ in terms of Schur functions.

THEOREM 1. *Let $f \in \mathscr{B}$ with the Schur functions $(f_n)_{n\geqslant 0}$. Then $|f| < 1$ a.e. on $\mathbb{T}$ (with respect to the Lebesgue measure m) if and only if*

$$\lim_n \int_{\mathbb{T}} |f_n|^2 \, dm = 0. \tag{2.1}$$

The following corollary is immediate from Theorem 1.

COROLLARY 2.1. *A function f in $\mathscr{B}$ is an exposed point if and only if*

$$\overline{\lim_n} \int_{\mathbb{T}} |f_n|^2 \, dm > 0.$$

Applying Fatou's theorem on nontangential limits [12, Chap. I, Sect. 5] to the real parts of (1.14), we obtain that

$$\sigma'(\zeta) = \frac{1 - |f(\zeta)|^2}{|1 - \zeta f(\zeta)|^2} \tag{2.2}$$

a.e. on $\mathbb{T}$. Here $\sigma' = d\sigma/dm$ is the Lebesgue derivative of σ.

In the theory of orthogonal polynomials, a measure σ with $\sigma' > 0$ a.e. on $\mathbb{T}$ is called an *Erdős measure.*

Since a non-zero function $1 - zf(z)$ in the Hardy algebra H^∞ cannot vanish on a subset of positive Lebesgue measure on $\mathbb{T}$ [12, Chap. II, Corollary 4.2], by the Amar–Fisher–Lederer theorem σ is an Erdős measure if and only if the Schur function f of σ (see (2.2)) is a non-exposed point of $\mathscr{B}$. This implies the following corollary.

COROLLARY 2.2. *A probability measure σ is an Erdős measure if and only if the Schur function f of σ satisfies* (2.1).

COROLLARY 2.3. *Let σ be a probability Erdős measure on $\mathbb{T}$ with Geronimus parameters $(a_n)_{n\geqslant 0}$. Then*

$$\lim_n a_n = 0.$$

Proof. By Corollary 2.2 the Schur function f of σ satisfies (2.1). By Geronimus' theorem [13] (see Sect. 1) the Schur parameters $(\gamma_n)_{n\geqslant 0}$ of f

satisfy $a_n = \gamma_n$, $n = 0, 1, 2, \ldots$. Using the mean-value property of holomorphic functions f_n and Cauchy's inequality, we obtain that

$$|a_n| = |\gamma_n| = |f_n(0)| = \left| \int_{\mathbb{T}} f_n \, dm \right| \leqslant \left(\int_{\mathbb{T}} |f_n|^2 \, dm \right)^{1/2},$$

which completes the proof since f satisfies (2.1). ∎

Corollary 2.3 is known as Rakhmanov's theorem [45, 46]. In view of the importance of Rakhmanov's theorem for the theory of orthogonal polynomials, serious efforts were undertaken to simplify the original proof. We mention papers by Máté *et al.* [35], by Rakhmanov [47], and by Nevai [41]. These efforts resulted in the extension of Szegő's theory to Erdős measures or even to measures with the Geronimus parameters $(a_n)_{n \geqslant 0}$ satisfying $\lim_n a_n = 0$ [36, 37, 39].

The class of probability measures with $\lim_n a_n = 0$ is called Nevai's class. By Rakhmanov's theorem Nevai's class contains Erdős' class. There are examples of pure jump measures [5, 31, 33], pure singular continuous measures [32], including some singular Riesz products [28], in Nevai's class. Totik [59] constructed further important examples of measures in Nevai's class. For any $\varepsilon > 0$ there exists a continuous function w with $m\{w > 0\} < \varepsilon$ such that $w \, dm$ belongs to Nevai's class. For any probability measure μ with support $\mathbb{T}$ there exists a probability measure σ in Nevai's class which is absolutely continuous with respect to μ.

In terms of orthogonal polynomials the difference between Nevai's and Erdős' classes is well demonstrated by the following beautiful results due to Nevai [41]:

$$\begin{aligned} \sigma' > 0 \text{ a.e.} &\Leftrightarrow \lim_n \sup_{l \geqslant 1} \int_{\mathbb{T}} \left| \frac{|\varphi_n|^2}{|\varphi_{n+l}|^2} - 1 \right| dm = 0, \\ \lim_n a_n = 0 &\Leftrightarrow \lim_n \inf_{l \geqslant 1} \int_{\mathbb{T}} \left| \frac{|\varphi_n|^2}{|\varphi_{n+l}|^2} - 1 \right| dm = 0, \end{aligned} \tag{2.3}$$

(see [29, Theorem B, p. 192; 35, Theorems 2 and 3, p. 64; 40, Theorem 1.1, p. 295; 41, Theorem 4, p. 325]). Clearly, Corollary 2.2 contributes one more equivalent condition to the first statement (2.3).

Following the philosophy presented in Sect. 1, Theorem 1 can be restated in terms of the convergence of continued fractions (1.7). Recall (see Sect. 1) that $(A_n/B_n)_{n \geqslant 0}$ are the approximants of (1.7). It is known [3, 54] (see also Sect. 4) that $\|A_n/B_n\|_\infty < 1$. It follows that $A_n(\zeta)/B_n(\zeta) \in \mathbb{D}$ for every

$\zeta \in \mathbb{T}$. Therefore we can calculate the pseudohyperbolic distance between $A_n(\zeta)/B_n(\zeta)$ and $f(\zeta)$. Recall [12, Chap. I, Sect. 1] that the pseudohyperbolic distance on $\mathbb{D}$ is defined by

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right| \tag{2.4}$$

and is extended to $\mathbb{T}$ by continuity.

Corollary 2.4. *Let $f \in \mathcal{B}$ and let $(A_n/B_n)_{n \geqslant 0}$ be the approximants of* (1.7). *Then f is a non-exposed point of $\mathcal{B}$ (equivalently σ is an Erdős measure) if and only if*

$$\lim_n \int_{\mathbb{T}} \rho^2(f, A_n/B_n)\, dm = 0. \tag{2.5}$$

Proof. By (1.4–1.5) we have

$$f(z) = \tau_0 \circ \tau_1 \circ \cdots \circ \tau_n(f_{n+1}), \qquad A_n/B_n = \tau_0 \circ \tau_1 \circ \cdots \circ \tau_n(0). \tag{2.6}$$

The pseudohyperbolic distance on $\mathbb{D}$ is invariant under *a* Möbius conformal isomorphism of $\mathbb{D}$ [12, Chap. I, Sect. 1]. Since for $z \in \mathbb{T}$ the Möbius transform $\tau_\kappa(w) = (zw + \gamma_\kappa) \cdot (1 + \bar{\gamma}_\kappa zw)^{-1}$ is a conformal isomorphism of $\mathbb{D}$, we obtain by (2.6) that $\rho(f, A_n/B_n) = |f_{n+1}|$ on $\mathbb{T}$. ∎

By Lebesgue's dominated convergence theorem [53, Chap. VII, Sect. 3, Theorem VIII.3.1] (2.5) is in fact equivalent to

$$\lim_n m\{\zeta \in \mathbb{T} : \rho(f, A_n/B_n) \geqslant \varepsilon\} = 0 \tag{2.7}$$

for every $\varepsilon > 0$. In other words $\rho(f, A_n/B_n)$ tends to zero in measure, notationally $\rho(f, A_n/B_n) \Rightarrow 0$.

Clearly, we can replace the pseudohyperbolic distance ρ in (2.7) by the Poincaré metric

$$P(z_1, z_2) = \log \frac{1 + \rho(z_1, z_2)}{1 - \rho(z_1, z_2)}.$$

Thus we obtain the following description of Erdős' class in terms of continued fractions and Lobachevskii's geometry.

Corollary 2.5. *A probability measure σ on $\mathbb{T}$ is an Erdős measure if and only if the Schur function f of σ satisfies*

$$P(f, A_n/B_n) \Rightarrow 0. \tag{2.8}$$

It is interesting to compare (2.8) with an analogous description of Szegő measures. Recall that a probability measure σ on $\mathbb{T}$ is called a Szegő measure if $\lim_n k_n < +\alpha$ (see (1.9)). By Szegő's theorem σ is a Szegő measure if and only if $\log \sigma' \in L^1(\mathbb{T})$ [51, Chap. XII, Sect. 12.3].

THEOREM 2.6. *A probability measure σ is a Szegő measure if and only if*

$$\lim_n \int_{\mathbb{T}} P(f, A_n/B_n)\, dm = 0. \tag{2.9}$$

Proof. Since $\rho(f, A_n/B_n) = |f_{n+1}|$ on $\mathbb{T}$, we obtain that

$$P(f, A_n/B_n) = \log \frac{1 + |f_{n+1}|}{1 - |f_{n+1}|}. \tag{2.10}$$

For $f \in \mathscr{B}$ we obviously have $\operatorname{Re}(1 - zf) > 0$ in $\mathbb{D}$. Hence $1 - zf$ is an outer function in H^∞ [12, Chap. II, Corollary 4.8a]. It follows that

$$\int_{\mathbb{T}} \log |1 - zf|^2\, dm = 0.$$

Combining this identity with (2.2), we obtain that

$$\int_{\mathbb{T}} \log \sigma'\, dm = \int_{\mathbb{T}} \log(1 - |f|^2)\, dm.$$

Next, by [12, Chap. V, Example 21(d)] (see also (6.1)) we have

$$\int_{\mathbb{T}} \log(1 - |f|^2)\, dm = \log \omega_n + \int_{\mathbb{T}} \log(1 - |f_{n+1}|^2)\, dm,$$

where

$$\omega_n = \prod_{\kappa=0}^{n} (1 - |\gamma_\kappa|^2) = \prod_{\kappa=0}^{n} (1 - |a_\kappa|^2) = \frac{1}{k_{n+1}^2}.$$

Since by Szegő's theorem [51]

$$\lim_n \log \frac{1}{k_{n+1}^2} = \int_{\mathbb{T}} \log \sigma'\, dm,$$

we conclude that σ is a Szegő measure if and only if

$$\lim_n \int_{\mathbb{T}} \log \frac{1}{1 - |f_{n+1}|^2}\, dm = 0. \tag{2.11}$$

If σ is a Szegő measure then (2.11) and the elementary inequality $\log \frac{1}{1-x} \geqslant x$ imply (2.1) and therefore (2.9) holds by (2.10). Similarly (2.9–2.10) imply (2.1) and (2.11). ∎

In [38] Máté *et al.* developed a method of weak and strong convergence. This method turned out to be very useful not only for the proof of Rakhmanov's theorem, but also for a deeper study of Erdős' class.

Roughly speaking this method is similar to a well-known method in the theory of quadratic forms. Suppose that we want to minimize a quadratic form Q in a Hilbert space over a hyperplane F. If we pick any sequence $(e_n)_{n \geqslant 0}$ of vectors in F such that $\lim_n Q(e_n) = \min$, then a priori we can only say that the sequence $(e_n)_{n \geqslant 0}$ converges to the extremal vector e in the weak topology. But if in addition we attract convexity arguments, such as the parallelogram identity, then we can conclude that in fact $(e_n)_{n \geqslant 0}$ converges to e in the strong topology.

Our proof of Theorem 1 also uses arguments of weak and strong convergence. The main difference is that we put this idea in the context of continued fractions (1.5), or, what is equivalent, in the context of Schur's algorithm.

Instead of the parallelogram identity, mentioned in the example above, we use the following formula which is interesting in itself.

THEOREM 2. *Let σ be a probability measure on $\mathbb{T}$ with infinite support, let $(\varphi_n)_{n \geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$, and set $b_n \stackrel{\text{def}}{=} \varphi_n / \varphi_n^*$. Let $(f_n)_{n \geqslant 0}$ be the Schur functions of the Schur function of σ. Then*

$$|\varphi_n|^2 \sigma' = \frac{1 - |f_n|^2}{|1 - \zeta b_n f_n|^2} \tag{2.12}$$

a.e. on $\mathbb{T}$.

Remark. Theorem 2 also holds for σ with finite support but in this case (2.12) is trivial. Clearly b_n is the finite Blaschke product constructed by the zeros of φ_n.

In Section 5 we present a simplified version of weak and strong arguments to provide a simple proof of Szegő's classical theorem. Combined with (2.12) this yields the following result.

THEOREM 2.5. *Let σ be a Szegő measure. Then*

$$\lim_n \int_{\mathbb{T}} \left| \log \frac{1}{|\varphi_n|^2} - \log \sigma' \right| dm = 0. \tag{2.13}$$

Our version of weak and strong arguments is based on two well-known theorems.

Let $M(\mathbb{T})$ be the Banach space of all finite Borel measures equipped with the variation norm. Let $C(\mathbb{T})$ be the Banach space of all continuous functions f on $\mathbb{T}$ with the standard sup-norm: $\|f\| = \|f\|_\infty = \sup\{|f(\zeta)|: \zeta \in \mathbb{T}\}$. By Riesz' theorem the conjugate space $C(\mathbb{T})^*$ is identified with $M(\mathbb{T})$ via the standard duality

$$(\mu, f) \to \int_{\mathbb{T}} f\, d\mu.$$

This duality determines the weak-$*$ topology in $M(\mathbb{T})$.

THEOREM (Helly's Theorem). *Let $(\mu_n)_{n \geqslant 0}$ be a sequence of finite non-negative Borel measures on $\mathbb{T}$. Then*

$$(*)\text{-}\lim_n \mu_n = \mu$$

if and only if for every open arc I on $\mathbb{T}$ with the endpoints carrying no point masses of μ we have

$$\lim_n \mu_n(I) = \mu(I).$$

Remark. It is important to observe that the necessity of Helly's theorem does not hold for real Borel measures. The sequence $\mu_n = \frac{1}{2}\delta_{\zeta_n} - \frac{1}{2}\delta_{\zeta_{n+1}}$, where $\zeta_n = \exp\{2\pi i \cdot \sum_{\kappa=1}^{n} 1/\kappa\}$, converges to zero in the weak-$(*)$ topology while $\mu_n(I) = \pm\frac{1}{2}$ infinitely often if I is any open arc on $\mathbb{T}$. This should be kept in mind in Section 5 in the proof of Szegő's theorem. See [48, Chap. III, Section 1, Sect. 55] for the proof of Helly's theorem in the stated form.

THEOREM (Jensen's Inequality). *Let (X, μ) be a probability space. Let v, $v \in L^1(\mu)$, be a real-valued function and φ a concave function on the real line $\mathbb{R}$. Then*

$$\int_X \varphi(v)\, d\mu \leqslant \varphi\left(\int_X v\, d\mu\right).$$

An elegant proof of Jensen's inequality can be found in [12, Chap. I, Section 6]. Theorem 2 can be generalized.

THEOREM 3. *Let σ be a probability measure on $\mathbb{T}$ with Schur function f. Let $(\varphi_n)_{n \geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$, $(f_n)_{n \geqslant 0}$ the Schur functions of f, $b_n = \varphi_n/\varphi_n^*$. Then*

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} |\varphi_n(\zeta)|^2\, d\sigma(\zeta) = \frac{1 + z f_n b_n}{1 - z f_n b_n}, \qquad z \in \mathbb{D}. \tag{2.14}$$

Theorem 3 has an interesting application to one important class of measures which we are going to describe. In [45, Lemma 2] Rakhmanov proved that

$$(*)\text{-}\lim_n |\varphi_n|^2\, d\sigma = dm \tag{2.15}$$

if σ is an Erdős measure. It was shown by Máté *et al.* [38, Corollary 2.2] that in fact Erdős measures satisfy

$$\lim_n \int_{\mathbb{T}} \left| |\varphi_n|^2\, \sigma' - 1 \right| dm = 0. \tag{2.16}$$

Later another proof of (2.16) was given by Rakhmanov [47]. In Section 6 we show how one can obtain (2.16) with the techniques developed for the proof of Theorem 1.

We say that a probability measure σ is a *Rakhmanov measure* if (2.15) holds.

It is clear from Theorem 3 that σ is a Rakhmanov measure if and only if the sequence $(f_n b_n)_{n\geqslant 0}$ converges to zero uniformly on compact subsets of $\mathbb{D}$:

$$f_n b_n \rightrightarrows 0 \tag{2.17}$$

In Section 7 we show that Rakhmanov measures can be described in terms of Geronimus parameters.

THEOREM 4. *Let σ be a probability measure on $\mathbb{T}$ with Geronimus parameters $(a_n)_{n\geqslant 0}$. Then σ is a Rakhmanov measure if and only if the sequence $(a_n)_{n\geqslant 0}$ satisfies the Máté–Nevai condition*

$$\lim_n a_n a_{n+\kappa} = 0 \tag{2.18}$$

for every κ, $\kappa = 1, 2, \ldots$.

The Máté–Nevai condition appeared first in [34] (for $\kappa = 1$) in relation to asymptotic properties of the ratio of orthogonal polynomials. In Section 7 we show that

$$\Phi^*_{n+1}/\Phi^*_n \rightrightarrows 1 \tag{2.19}$$

in $\mathbb{D}$ if and only if the Geronimus parameters of σ satisfy the Máté–Nevai condition (2.18) for every, κ, $\kappa = 1, 2, \ldots$. Here $\Phi_n \overset{\text{def}}{=} k_n^{-1} \cdot \varphi_n$ stands for a monic orthogonal polynomial.

A simple analysis of (2.18) (see Sect. 7) shows that

$$\lim_n \frac{\operatorname{Card}\{j\colon |a_j| \geqslant \varepsilon,\ j \leqslant n\}}{n} = 0 \tag{2.20}$$

for every positive ε, if $(a_n)_{n \geqslant 0}$ satisfies (2.18) for every κ, $\kappa = 1, 2, \ldots$. It is well know that the latter condition is equivalent to

$$\lim_n \frac{1}{n} \sum_{\kappa=0}^{n-1} |a_\kappa| = 0. \tag{2.21}$$

Of course, there are sequences satisfying (2.21) and not satisfying the Máté–Nevai condition for every κ (see Sect. 7).

It follows from the definition that Rakhmanov measures cannot vanish on any open arc of $\mathbb{T}$. Therefore $\operatorname{supp}(\sigma) = \mathbb{T}$ for any Rakhmanov measure σ. In [13, 14] Geronimus proved that $\operatorname{supp}(\sigma) = \mathbb{T}$ for any probability measure σ with the parameters satisfying

$$\overline{\lim_n} \sqrt[n]{\prod_{\kappa=0}^{n-1} (1 - |a_\kappa|^2)} = 1. \tag{2.22}$$

It is easy to construct an example of a sequence $(a_n)_{n \geqslant 0}$ satisfying (2.18) for κ, $\kappa = 1, 2, \ldots$ but not (2.22).

Let us turn back to Corollary 2.5. It is natural to ask: what happens if we replace the Poincaré metric in (2.8) with Euclidian metric? The answer is given by the following theorem. Recall that a function f in $\mathscr{B}$ is called *inner* if $|f| = 1$ a.e. on $\mathbb{T}$ [12, Chap. II, Sect. 6].

THEOREM 5. *Let f be a function in $\mathscr{B}$ with Schur parameters $(\gamma_n)_{n \geqslant 0}$ and let $(A_n)_{n \geqslant 0}$, $(B_n)_{n \geqslant 0}$ be the corresponding Wall polynomials. Then*

$$\lim_n \int_{\mathbb{T}} \left| f - \frac{A_n}{B_n} \right|^2 dm = 0 \tag{2.23}$$

if and only if either f is an inner function or $\lim_n \gamma_n = 0$.

We also prove that the Schur functions of Rakhmanov measures satisfy (2.23).

THEOREM 6. *Let σ be a Rakhmanov measure and let f be the Schur function of σ. Then*

$$\lim_n \int_{\mathbb{T}} \left| f - \frac{A_n}{B_n} \right|^2 dm = 0.$$

The combination of Theorems 5 and 6 yields a curious application to Rakhmanov measures.

COROLLARY 2.6. *Let σ be a Rakhmanov measure which does not belong to Nevai's class. Then σ is a singular measure.*

Proof. If σ is a Rakhmanov measure then by Theorem 6 the Schur function f of σ satisfies (2.23). By Theorem 5 either f is an inner function or $\lim \gamma_n = 0$. The second possibility is excluded by the assumption that σ does not belong to Nevai's class and by Geronimus' theorem. It follows that f is an inner function and therefore σ is a singular measure (see (2.2)). ▮

Geronimus' theorem and (1.16) lead to another interesting application in the theory of orthogonal polynomials. Recall [12, Chap. II, Sect. 1] that the Hardy class H^p, $0 < p < \infty$, consists of all holomorphic functions f in $\mathbb{D}$ satisfying

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta) = \|f\|_p^p < +\infty.$$

We identify f, $f \in H^p$, with the boundary values $\lim_{r \to 1-0} f(r\zeta)$ of f on $\mathbb{T}$. By Smirnov's theorem [12, Chap. III, Sect. 2, Theorem 2.4] the Herglotz transform F_σ (see (1.12)) of any probability measure σ belongs to $\bigcap_{p<1} H^p$.

THEOREM 7. *Given a probability measure σ on $\mathbb{T}$*

$$\frac{\psi_n^*}{\varphi_n^*} \Rightarrow F_\sigma \tag{2.24}$$

if and only if either σ is a singular measure or σ is in Nevai's class.

In case (2.24) holds the convergence in (2.24) takes place in the metric of $L^p(\mathbb{T})$, $0 < p < 1$.

THEOREM 8. *Let σ be either a singular measure or a measure in Nevai's class. Then for every p, $0 < p < 1$,*

$$\lim_n \int_{\mathbb{T}} \left| \frac{\psi_n^*}{\varphi_n^*} - F_\sigma \right|^p dm = 0. \tag{2.25}$$

It is interesting to compare Theorem 5 with known results in the theory of continued fractions. To begin with we observe that any continued fraction (1.7) is a T-fraction [23], i.e., a continued fraction of the form

$$\mathop{\mathrm{K}}_{n=1}^{\infty} [F_n z/(1+G_n z)], \tag{2.26}$$

which converges to a meromorphic function in $\mathbb{C}$ if

$$\lim_n F_n = \lim_n G_n = 0; \tag{2.27}$$

see Theorem 7.23 of [23]. In the case of (1.7) these conditions are equivalent to

$$\lim_n \gamma_n/\gamma_{n-1} = 0. \tag{2.28}$$

Therefore Theorem 5 follows from Theorem 7.23 of [23] if the sequence of Schur parameters decays to zero faster any sequence of exponentials.

In Section 8 we prove (see Lemma 8.2) that the convergence in measure of the Wall approximants A_n/B_n on any subset of positive Lebesgue measure of the Lebesgue support $E(\sigma)$ of σ implies that σ is in Nevai's class. This result can be applied to the study of gaps in the continuous spectrum of σ for σ satisfying $\overline{\lim}_n |\gamma_n| > 0$. Indeed, if we can prove that A_n/B_n converges on an open arc of $\mathbb{T}$, then by Lemma 8.2 and $\overline{\lim}_n |\gamma_n| > 0$ we conclude that the Schur function of σ is unimodular on this open arc and therefore $\sigma' \equiv 0$ on it.

The simplest example of this sort is given by σ with constant Geronimus parameters $a_n \equiv a$, $n = 0, 1, ...,$ $0 < |a| < 1$. Clearly,

$$f(z) = \frac{a}{1-} \, \frac{(1-|a|^2)\, z}{1+z} \, - \cdots - \frac{(1-|a^2|)\, z}{1+z} \, - \cdots \tag{2.29}$$

is the Schur function of σ. The orthogonal polynomials in $L^2(d\sigma)$ are called Geronimus polynomials (see details in the recent paper [19]). The Schur function f satisfies the equation (see (1.3))

$$f(z) = \frac{zf(z)+a}{1+\bar{a}zf(z)}, \qquad z \in \mathbb{D},$$

which implies that $|f| = 1$ exactly on the arc $I_a = \{e^{i\theta} \colon |\sin \frac{\theta}{2}| \leqslant |a|\}$. The convergence of (2.29) on I_a follows from the convergence theorem for periodic continued fractions [23, Theorem 3.2] or from Worpitsky's theorem [23, Corollary 4.36 (B)].

THEOREM (J. Worpitsky). *A continued fraction $K(a_n/1)$ converges to a finite value if*

$$|a_n| \leqslant 1/4, \qquad n=1, 2, ... \tag{2.30}$$

We apply Worpitsky's theorem to the continued fraction $K(a_n(z)/1)$ with $a_n(z)=(1-|a|^2)\, z(1+z)^{-2}$, $n=1, 2, ...$, which is equivalent to (2.29). For $e^{i\theta} \in I_a$ we have

$$|a_n(e^{i\theta})| = \frac{1-|a|^2}{4\cos^2(\theta/2)} \leqslant 1/4.$$

Since $|f|=1$ exactly on I_a, we conclude by Corollary 8.4 and Worpitsky's theorem that the Wall approximants for f converge only on I_a.

THEOREM (A. Pringsheim). *A continued fraction $\mathsf{K}_{n=1}^{\infty}(a_n/b_n)$ converges to a finite value if*

$$|b_n| \geqslant |a_n|+1, \qquad n=1, 2, ... \tag{2.31}$$

If r_n is the nth approximant of $\mathsf{K}(a_n/b_n)$, then $|r_n|<1$, $n=1, 2, ...$

See [23, Theorem 4.35] for a proof of Pringsheim's theorem. In Section 9 we combine Pringsheim's theorem with the approach described above to obtain the following result on a gap in the spectrum.

THEOREM 9. *Let σ be a probability measure on $\mathbb{T}$ with Geronimus parameters $(\gamma_n)_{n\geqslant 0}$ satisfying*

$$\varliminf_n |\gamma_n| > 0; \tag{2.32.1}$$

$$\lim_n \arg(\bar{\gamma}_n \gamma_{n-1}) = \theta, \qquad \theta \in \mathbb{R}. \tag{2.32.2}$$

Then there exists an open arc I on $\mathbb{T}$ centered at $\exp(i\theta)$ such that $\operatorname{supp}(\sigma) \cap I$ is a finite set.

Clearly, Theorem 9 is in good agreement with the example of Geronimus polynomials. It is also useful to compare Theorem 9 with Stieltjes' well-known theorem.

THEOREM (Stieltjes [50]). *Let σ be a probability measure on $\mathbb{T}$ with real Geronimus parameters $(\gamma_n)_{n\geqslant 0}$ satisfying*

$$\lim_n (1+\gamma_n)(1-\gamma_{n+1}) = 0. \tag{2.33}$$

Then the derived set of $\operatorname{supp}(\sigma)$ is $\{-1\}$.

It is easy to see [18, p. 407] that for real sequences (2.33) is equivalent to (2.32.2) with $\theta = 0$ and $\lim |\gamma_n| = 1$.

The following result was obtained in [18] (Theorem 6, (i) $\Leftrightarrow$ (vi)).

THEOREM 2.7. *Let σ be a probability measure on $\mathbb{T}$ with infinite support. Then the following statements are equivalent*:

(1) *the derived set of* $\mathrm{supp}(\sigma)$ *is* $\{\tau\}$;

(2) $-\lim_n \bar{\gamma}_n \gamma_{n-1} = \tau$.

In Section 9 we prove the following extension of Theorem 2.7.

THEOREM 10. *Let σ be a probability measure on $\mathbb{T}$ with Geronimus parameters $(\gamma_n)_{n \geqslant 0}$ satisfying*

$$\lim_n |\gamma_n| = 1. \tag{2.34}$$

Then the following statements are equivalent:

(1) τ, $\tau \in \mathbb{T}$, *is in the derived set of* $\mathrm{supp}(\sigma)$;

(2) *there exists an infinite subset Λ of the set of positive integers such that* $-\lim_{n \in \Lambda} \bar{\gamma}_n \gamma_{n-1} = \tau$.

Remark. This theorem was obtained independently by L. Golinskii [58, Theorem 5] by a different method.

Proof of Theorem 2.7. (1) $\Rightarrow$ (2). The part (i) $\Rightarrow$ (ii) of Theorem 6 of [18] yields $(*)$-$\lim_n |\varphi_n|^2 \, d\sigma = \delta_\tau$. Since $f_n b_n$ is the Schur function of σ by Theorem 5 and since $f_n(0)\, b_n(0) = -\gamma_n \bar{\gamma}_{n-1}$ we obtain (2).

(2) $\Rightarrow$ (1). Clearly, (2) implies (2.34) and the result follows by Theorem 10. ∎

In Section 10 we describe absolutely continuous probability measures with smooth positive densities in terms of the decrease of their Schur functions. We define the Hölder class as follows. For $0 < \alpha < 1$ we put

$$\Lambda_\alpha = \{ f \in C(\mathbb{T}) : |f(e^{i(x+t)}) - f(e^{ix})| \leqslant C_f \, |t|^\alpha, \, x, t \in \mathbb{R} \}.$$

For $\alpha = 1$ we denote by Λ_1 the Zygmund class

$$\Lambda_1 = \{ f \in C(\mathbb{T}) : |f(e^{i(x+t)}) + f(e^{i(x-t)}) - 2f(e^{ix})| \leqslant C_f \cdot |t|, \, x, t \in \mathbb{R} \}.$$

Now, let $n < \alpha \leqslant n+1$, where n is a positive integer. Then Λ_α denotes the space of all functions f on $\mathbb{T}$ with the nth derivative $f^{(n)}$ in $\Lambda_{\alpha - n}$.

THEOREM 11. *Let σ be a probability measure on $\mathbb{T}$ with Schur functions $(f_n)_{n\geqslant 0}$ satisfying*

$$\|f_n\|_\infty = O\left(\frac{1}{n^\alpha}\right), \qquad \alpha > 0. \tag{2.35}$$

Then σ is absolutely continuous and $(\sigma')^{-1} \in \Lambda_\alpha$.

THEOREM 12. *Let σ be an absolutely continuous probability measure with $(\sigma')^{-1} \in \Lambda_\alpha$, and let $(f_n)_{n\geqslant 0}$ be the Schur functions of σ. Then*

$$\|f_n\|_\infty = O\left(\frac{\log n}{n^\alpha}\right). \tag{2.36}$$

COROLLARY 2.8. *Let σ be an absolutely continuous probability measure with $(\sigma')^{-1} \in \Lambda_\alpha$. Then the Geronimus parameters $(a_n)_{n\geqslant 0}$ of σ satisfy*

$$a_n = O\left(\frac{\log n}{n^\alpha}\right). \tag{2.37}$$

Proof. This follows from the elementary inequality

$$|a_n| = \left|\int_{\mathbb{T}} f_n \, dm\right| \leqslant \|f_n\|_\infty. \quad \blacksquare$$

For $0 < \alpha < 1$ Corollary 2.8 was proved in [27]. See [17] for the general case $\alpha > 0$.

The following corollaries, which are immediate from Theorem 11 and Theorem 12 by (4.22), demonstrate a remarkable similarity in the behaviour of A_n/B_n and of the partial Fourier sums of f; see [56, Chap. II, Theorem 10.8].

COROLLARY 2.9. *Let σ be an absolutely continuous measure with $(\sigma')^{-1} \in \Lambda_\alpha$. Let A_n, B_n be the corresponding Wall polynomials. Then*

$$\left\| f - \frac{A_n}{B_n} \right\|_\infty = O\left(\frac{\log n}{n^\alpha}\right), \qquad n \to +\infty. \tag{2.38}$$

COROLLARY 2.10. *Let σ be a probability measure on $\mathbb{T}$ with Schur function f satisfying $\|f\|_\infty < 1$. Suppose that the Wall approximants A_n/B_n satisfy*

$$\left\| f - \frac{A_n}{B_n} \right\|_\infty = O\left(\frac{1}{n^\alpha}\right), \qquad n \to +\infty, \quad \alpha > 0. \tag{2.39}$$

Then σ is absolutely continuous and $(\sigma')^{-1}, f \in \Lambda_\alpha$.

Proof. By (4.15) and by (2.39)

$$\frac{\omega_n}{|B_n|^2} = 1 - \left|\frac{A_n}{B_n}\right|^2 \rightrightarrows 1 - |f|^2$$

uniformly on $\mathbb{T}$, and therefore $\sup_{\mathbb{T}} ||f| - |A_n^*/B_n|| \to 0$, $n \to +\infty$. It follows from (4.22) that

$$\|f_n\|_\infty = O\left(\frac{1}{n^\alpha}\right), \qquad n \to +\infty,$$

which completes the proof by Theorem 11. ▮

It is interesting to compare Corollary 2.10 with Bernstein's theorem [2]. The inverse problem of approximation by rational functions in the uniform norm was first considered by Gonchar [20] who discovered the essential difference from the polynomial case. The best possible result in this direction is due to Y. Brudnyi [4].

Let $\mathrm{Lip}(\alpha, p) = \{f \in L^p(\mathbb{T}) : \|\Delta_h^\kappa f\|_{L^p} \leqslant \mathrm{cont}\, |h|^\alpha\}$. Here κ is the smallest integer satisfying $\kappa > \alpha$ and $\Delta_h^\kappa = \Delta_h \Delta_h^{\kappa-1}$, $\Delta_h f = f(e^{i(x+h)}) - f(e^{ix})$.

THEOREM (Yu. Brudnyi [4]). *Let $\mathscr{R}_n$ be the set of all rational functions of order not exceeding n. Then*

$$\mathrm{Lip}\left(\alpha, \frac{1}{\alpha} + \varepsilon\right) \subset \left\{f\colon \mathrm{dist}_{L^\infty}(f, \mathscr{R}_n) = O\left(\frac{1}{n^\alpha}\right)\right\} \subset \mathrm{Lip}\left(\alpha, \frac{1}{\alpha} - \varepsilon\right).$$

Thus, Corollary 2.10 shows that for smooth f the Wall approximants behave like polynomials rather than like general rational fractions.

Basic Notations

m	the normalized ($\int_{\mathbb{T}} dm = 1$) Lebesgue measure on $\mathbb{T} \overset{\text{def}}{=} \{\zeta \in \mathbb{C} : |\zeta| = 1\}$.
$C(\mathbb{T})$	the Banach algebra of all continuous functions on $\mathbb{T}$(see Section 2).
$M(\mathbb{T})$	the Banach space of all finite Borel measures (see Section 2).
$\mathscr{B}$	the unit ball of the Hardy algebra H^∞ (see Section 1).

f_n	the nth Schur function of a function f, $f \in \mathscr{B}$ (see Section 1).
γ_n	the nth Schur parameter of f, $f \in \mathscr{B}$ (see Section 1).
$\mathscr{P}_n$	the set of polynomials in z of degree $\leqslant n$ (see Section 1).
p^*	the adjoint polynomial for p, $p \in \mathscr{P}_n$ (see (1.10)).
A_n and B_n	the Wall polynomials (see Section 1, (1.6)).
a_n	the nth Geronimus parameter of a probability measure σ, $\sigma \in M(\mathbb{T})$ (see Section 1).
φ_n	the nth orthogonal polynomial in $L^2(d\sigma)$.
$b_n \stackrel{\text{def}}{=} \varphi_n/\varphi_n^*$	the finite Blaschke product with the zeros of φ_n (see Section 2).
$\Phi_n = k_n^{-1} \cdot \varphi_n$	the monic orthogonal polynomial of order n (see Section 2).
ψ_n	the nth orthogonal polynomial corresponding to the parameters $(-a_n)_{n \geqslant 0}$.
F_σ	the Herglotz transform of a probability measure σ.
H^p	the Hardy class (see Section 2).
$(*)$	weak topology is the topology of a locally convex linear space X induced by the standard duality of X and its pre-dual space Y. The choices $X = M(\mathbb{T})$, $Y = C(\mathbb{T})$; $X = L^\infty(\mathbb{T})$, $Y = L^1(\mathbb{T})$ are especially important for the present paper.
$E(\sigma) = \{\zeta \in \mathbb{T} : \sigma'(\zeta) = d\sigma/dm(\zeta) > 0\}$	the Lebesgue support of σ.

3. CONTINUED FRACTIONS

Let $(p_n)_{n \geqslant 1}$ and $(q_n)_{n \geqslant 0}$ be sequences of complex numbers. A continued fraction

$$q_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} (p_n/q_n) = q_0 + \frac{p_1}{q_1} \underset{+}{} \frac{p_2}{q_2} \underset{+}{} \cdots \underset{+}{} \frac{p_n}{q_n} \underset{+}{} \cdots \tag{3.1}$$

is an algorithm which determines a sequence of approximants $(P_n/Q_n)_{n\geqslant 0}$. According to this algorithm the approximant P_n/Q_n is obtained by interrupting the infinite decomposition (3.1) at the term p_n/q_n and making all arithmetic operations without cancellations.

The definition of a continued fraction presented not only uniquely determines the values of the quotients P_n/Q_n but also determines the numerators P_n and the denominators Q_n. Elementary induction shows that P_n and Q_n satisfy Euler's recurrence formulae

$$\begin{aligned} P_n &= q_n P_{n-1} + p_n P_{n-2}, \\ Q_n &= q_n Q_{n-1} + p_n Q_{n-2}, \qquad n=1,2,..., \end{aligned} \tag{3.2}$$

understanding that

$$P_{-1}=1, \qquad P_0=q_0, \qquad Q_{-1}=0, \qquad Q_0=1. \tag{3.3}$$

The numbers p_n and q_n are called the *partial numerators* and the *partial denominators* of a continued fraction. Very often Euler's recurrence formulae (3.2) and (3.3) are used as a definition of a continued fraction.

To make a continued fraction algorithm more transparent we consider the Möbius transforms

$$s_n(w) = \frac{p_n}{w+q_n}, \qquad n=1,2,...,$$

of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For $n=0$ we put $s_0(w) = w + q_0$. Then the superposition $S_n(w) = s_0 \circ s_1 \circ \cdots \circ s_n(w)$ is a Möbius transform and

$$\begin{aligned} \frac{P_n}{Q_n} &= S_n(0) = s_0 \circ s_1 \circ \cdots \circ s_n(0) \\ &= s_0 \circ s_1 \circ \cdots \circ s_{n+1}(\infty) = S_{n+1}(\infty). \end{aligned} \tag{3.4}$$

The following theorem is well known [23, Theorem 2.1, Sect. 2.1]. However, we provide a proof which allows us to obtain two important formulae.

THEOREM 3.1. *Let P_n be the nth numerator and let Q_n be the nth denominator of a continued fraction $q_0 + \mathrm{K}_{n=1}^{\infty}(p_n/q_n)$. Then*

$$\begin{aligned} S_n(w) &= \frac{P_{n-1}w + P_n}{Q_{n-1}w + Q_n}, \\ P_n Q_{n-1} - P_{n-1} Q_n &= (-1)^{n-1}\, p_1 \cdots p_n. \end{aligned} \tag{3.5}$$

Proof. Euler's recurrence formulae (3.2) and (3.3) can be put into matrix form as follows:

$$\begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} = \begin{pmatrix} 1 & q_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & p_1 \\ 1 & q_1 \end{pmatrix}\cdots\begin{pmatrix} 0 & p_n \\ 1 & q_n \end{pmatrix}, \qquad n=0, 1, ... \tag{3.6}$$

Identifying the Möbius transforms s_κ with the corresponding (2×2)-matrices and applying (3.6) to a column $(w, 1)$, we obtain the first formula (3.5). Applying the multiplicative functional $C \mapsto \det(C)$ to (3.6), $\det(C)$ being the determinant of a matrix C, we obtain the second formula (3.5). ∎

Corollary 3.2. *For any continued fraction $q_0 + \mathrm{K}_{n=1}^{\infty}(p_n/q_n)$ we have*

$$\frac{P_n}{P_{n-1}} = q_n + \frac{p_n}{q_{n-1}} + \frac{p_{n-1}}{q_{n-2}} + \cdots \frac{p_2}{+q_1} \frac{p_1}{+q_0} \tag{3.7}$$

$$\frac{Q_n}{Q_{n-1}} = q_n + \frac{p_n}{q_{n-1}} + \frac{p_{n-1}}{q_{n-2}} + \cdots \frac{p_2}{+q_1}. \tag{3.8}$$

Proof. Applying the operation of transposition to the matrix identity (3.6), we obtain

$$\begin{pmatrix} P_{n-1} & Q_{n-1} \\ P_n & Q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p_n & q_n \end{pmatrix}\cdots\begin{pmatrix} 0 & 1 \\ p_1 & q_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q_0 & 1 \end{pmatrix}. \tag{3.9}$$

It is clear that (3.9) can be presented as a superposition of Möbius transforms

$$\frac{P_{n-1}w + Q_{n-1}}{P_n w + Q_n} = t_n \circ \cdots \circ t_1 \circ t_0(w), \tag{3.10}$$

where $t_0(w) = w/(q_0 w + 1)$, $t_\kappa(w) = 1/(p_\kappa w + q_\kappa)$, $\kappa = 1, 2, ...$ Now, notice that (3.7) is (3.10) for $w = \infty$ and (3.8) is (3.10) for $w = 0$. ∎

Formulae (3.7) and (3.8) are used in the convergence theory of continued fractions [23, Chap. 4, (4.1.3)]. We apply them for calculating Schur parameters for some important functions. Also, they can be used to obtain formulae for P_n, Q_n.

We suppose that $p_n \neq 0$, $n = 1, 2, ...$ Then, by Theorem 3.1, S_n is a homeomorphism of the Riemann sphere. It follows that for every K in $\hat{\mathbb{C}}$ the equation $K = S_n(w)$ has a unique solution $w_n = w_n(K)$.

LEMMA 3.3. *Let* $p_\kappa \neq 0$, $\kappa = 1, 2, ..., n$, $K \in \hat{\mathbb{C}}$ *and* $w_\kappa = S_\kappa^{-1}(K)$, $\kappa = 1, ..., n$. *Then*

$$P_n + P_{n-1} w_n = \prod_{\kappa=0}^{n} (q_\kappa + w_\kappa)$$
$$Q_n + Q_{n-1} w_n = \prod_{\kappa=1}^{n} (q_\kappa + w_\kappa). \tag{3.11}$$

Proof. Observing that $s_\kappa(w_\kappa) = w_{\kappa-1}$, by (3.2) we obtain that

$$\begin{aligned} P_n + P_{n-1} w_n &= (q_n + w_n)\, P_{n-1} + p_n P_{n-2} \\ &= (q_n + w_n)(P_{n-1} + P_{n-2} w_{n-1}) = \cdots \\ &= (q_n + w_n) \cdots (P_0 + P_{-1} w_0) = \prod_{\kappa=0}^{n} (q_\kappa + w_\kappa). \end{aligned}$$

The second identity (3.11) is obtained similarly. ∎

THEOREM 3.4. *Let* $p_\kappa \neq 0$, $\kappa = 1, 2, ..., n$, $K \in \hat{\mathbb{C}}$, $w_\kappa = S_\kappa^{-1}(K)$, $\kappa = 1, ..., n$. *Then*

$$P_n = \left\{ \frac{1}{1+} \frac{w_n}{q_n} \frac{p_n}{+q_{n-1}+} \cdots \frac{p_2}{+q_1} \frac{p_1}{+q_0} \right\} \cdot \prod_{\kappa=0}^{n} (q_\kappa + w_\kappa).$$
$$Q_n = \left\{ \frac{1}{1+} \frac{w_n}{q_n} \frac{p_n}{+q_{n-1}+} \cdots \frac{p_2}{+q_1} \right\} \cdot \prod_{\kappa=1}^{n} (q_\kappa + w_\kappa). \tag{3.12}$$

Proof. Combining the first identity (3.11) with (3.7), we obtain the first identity (3.12). Similarly, the second identity (3.12) follows from (3.8) and the second identity (3.11). ∎

On the Riemann sphere $\hat{\mathbb{C}}$, we consider the metric [7, Chap. I, Sect. 1]

$$k(w_1, w_2) = \frac{|w_1 - w_2|}{\sqrt{1 + |w_1|^2} \cdot \sqrt{1 + |w_2|^2}}, \tag{3.13}$$

which is equivalent to the Euclidean metric $|w_1 - w_2|$ on any compact subset of $\mathbb{C}$. It is easy to prove [7, Chap. X, Sect. 6] that the metric k is invariant under the transforms

$$w = \frac{1 + z\bar{a}}{z - a}, \qquad a \in \hat{\mathbb{C}}, \tag{3.14}$$

which correspond to rotations of the Riemann sphere.

DEFINITION. A continued fraction $q_0 + \mathsf{K}_{n=1}^{\infty}(p_n/q_n)$ is said to converge to K, $K \in \hat{\mathbb{C}}$, if

$$\lim_n k(P_n/Q_n, K) = 0$$

It is easy to see that if, say, $p_{n+1} = 0$, then all approximants P_m/Q_m for $m > n$ coincide with P_n/Q_n (see (3.2) or (3.4)) and therefore this continued fraction converges.

If p_n and q_n are complex functions on some set X, then the approximants P_n/Q_n are functions with values in $\hat{\mathbb{C}}$ defined on the same set X. In what follows, to specify the character of convergence of the approximants P_n/Q_n on X we apply the corresponding terminology to the continued fraction directly. However, we distinguish some exceptional cases where the specific terminology for continued fractions is used. So, we say that a continued fraction converges absolutely if

$$\left|\frac{P_0}{Q_0}\right| + \sum_{n=0}^{\infty}\left|\frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n}\right| < +\infty.$$

Also, we say that a continued fraction converges unconditionally if for every n the continued fraction $\mathsf{K}_{\kappa=n}^{\infty}(p_\kappa/q_\kappa)$ converges to a finite value.

Now, let us consider the continued fractions (1.7). As has already been observed, the Wall polynomials A_n, B_n are the numerators and denominators of (1.7). If, say, $\gamma_{n+1} = 0$, then the corresponding continued fraction does not make sense. However, as we show later in Sect. 4, A_{n+1} and B_{n+1} can be defined by (4.5), which implies that $A_{n+1} = A_n$, $B_{n+1} = B_n$. This explains why (1.7) fails. The reason is that for $\gamma_{n+1} = 0$ we obtain two identical approximants A_n/B_n and A_{n+1}/B_{n+1}. To exclude an excessive approximant one should eliminate the corresponding part of (1.7). Suppose first that all γ_n's are non-zero. Then

$$-\frac{(1-|\gamma_n|^2)(\gamma_{n+1}/\gamma_n)\,z}{1+(\gamma_{n+1}/\gamma_n)\,z-(1-|\gamma_{n+1}|^2)(\gamma_{n+2}/\gamma_{n+1})\,z/(1+(\gamma_{n+2}/\gamma_{n+1})\,z+w)}$$

$$= -\frac{(1-|\gamma_n|^2)(\gamma_{n+1}/\gamma_n+\gamma_{n+2}z^2/\gamma_n+\gamma_{n+1}zw/\gamma_n)}{1+(\gamma_{n+1}/\gamma_n)\,z+(\gamma_{n+2}/\gamma_n)\,z^2+(\gamma_{n+1}/\gamma_n)\,zw+\bar{\gamma}_{n+1}\gamma_{n+2}z+w}$$

$$\to -\frac{(1-|\gamma_n|^2)(\gamma_{n+2}/\gamma_n)\,z^2}{1+(\gamma_{n+2}/\gamma_n)\,z^2+w}$$

as $\gamma_{n+1} \to 0$. This shows how one can exclude indefinite terms in (1.7) corresponding to zero parameters. In what follows, we do not specify this agreement explicitly assuming that the corresponding adjustment is made. However, this construction can be avoided if we agree in defining (1.7) by the recurrence formulae (4.5).

4. WALL'S POLYNOMIALS

Now, we apply the theory presented in Section 3 to the study of the Wall continued fraction (1.5) of Schur's algorithm. Euler's formulae (3.2) for this fraction take the form

$$\begin{aligned} P_{2n} &= \gamma_n P_{2n-1} + P_{2n-2}, \\ Q_{2n} &= \gamma_n Q_{2n-1} + Q_{2n-2}, \qquad n = 1, 2, ..., \end{aligned} \tag{4.1.1}$$

$$\begin{aligned} P_{2n+1} &= z\bar{\gamma}_n P_{2n} + z(1 - |\gamma_n|^2)\, P_{2n-1}, \\ Q_{2n+1} &= z\bar{\gamma}_n Q_{2n} + z(1 - |\gamma_n|^2)\, Q_{2n-1}, \qquad n = 0, 1, ..., \end{aligned} \tag{4.1.2}$$

where $P_{-1} = 1$, $P_0 = \gamma_0$, $Q_{-1} = 0$, $Q_0 = 1$. This is immediate from (3.2) if we notice that

$$p_{2n} = 1, \qquad q_{2n} = \gamma_n, \qquad p_{2n+1} = z(1 - |\gamma_n|^2), \qquad q_{2n+1} = z\bar{\gamma}_n;$$

see (1.5).

Recall (see Section 1) that $A_n \overset{\text{def}}{=} P_{2n}$ and $B_n \overset{\text{def}}{=} Q_{2n}$ are called the Wall polynomials associated with the Schur parameters $(\gamma_n)_{n \geqslant 0}$. Since $\deg P_{2n} = \deg P_{2n-1}$ and $\deg Q_{2n} = \deg Q_{2n-1}$ by (4.1.1), it is clear that A_n, $B_n \in \mathcal{P}_n$.

The following simple lemma shows that Wall polynomials A_n, B_n uniquely determine the Wall continued fraction (1.5) of the corresponding Schur algorithm.

Lemma 4.1. *For $n = 0, 1, ...$ we have*

$$P_{2n+1} = zB_n^*, \qquad Q_{2n+1} = zA_n^*. \tag{4.2}$$

Proof. For $n = 0$ we have by (4.1.2)

$$P_1 = z\,|\gamma_0|^2 + z(1 - |\gamma_0|^2) = z = zB_0^*, \qquad Q_1 = z\bar{\gamma}_0 = zA_0^*.$$

Assuming now that (4.2) holds for all indices smaller than n and observing that $\deg P_{2n} = \deg P_{2n-1} = \deg A_n$, we obtain by (4.1.1.–4.1.2) that

$$\begin{aligned} zQ_{2n}^* &= z\{\bar{\gamma}_n Q_{2n-1}^* + zQ_{2n-2}^*\} = z\{\bar{\gamma}_n P_{2n-2} + P_{2n-1}\} \\ &= z\{\bar{\gamma}_n P_{2n} - |\gamma_n|^2\, P_{2n-1} + P_{2n-1}\} \\ &= z\bar{\gamma}_n P_{2n} + z(1 - |\gamma_n|^2)\, P_{2n-1} = P_{2n+1}. \end{aligned}$$

Similarly, $Q_{2n+1} = zP_{2n}^*$. ∎

The sequence $(A_n/B_n)_{n\geqslant 0}$ corresponds to the even part of Wall's continued fraction, while $(zB_n^*/zA_n^*)_{n\geqslant 0}$ corresponds to the odd part of (1.5). The following theorem can be proved as a consequence of general formulae [23, Chap. 2, (2.4.24), (2.4.29)]. However, we provide a proof which follows the arguments of [26]. This allows us to present important recurrence formulae for Wall polynomials.

THEOREM 4.2. *The sequence* 1/0, 0/1, A_0/B_0, ..., A_n/B_n, ... *is the sequence of approximants of the continued fraction*

$$W_{\text{even}} = \frac{\gamma_0}{1} - \frac{(1-|\gamma_0|^2)(\gamma_1/\gamma_0)\,z}{1+(\gamma_1/\gamma_0)\,z} - \cdots - \frac{(1-|\gamma_{n-1}|^2)(\gamma_n/\gamma_{n-1})\,z}{1+(\gamma_n/\gamma_{n-1})\,z} - \cdots. \tag{4.3}$$

The sequence 1/0, 0/1, zB_0^*/zA_0^*, ..., zB_n^*/zA_n^*, ... *is the sequence of approximants of the continued fraction*

$$W_{\text{odd}} = \frac{z}{\bar\gamma_0 z} + \frac{(1-|\gamma_0|^2)\,\bar\gamma_1 z}{\gamma_0\bar\gamma_1 + z} - \frac{(1-|\gamma_1|^2)(\bar\gamma_2/\bar\gamma_1)\,z}{(\bar\gamma_2/\bar\gamma_1)+z} - \cdots - \frac{(1-|\gamma_{n-1}|^2)(\bar\gamma_n/\bar\gamma_{n-1})\,z}{(\bar\gamma_n/\bar\gamma_{n-1})+z} - \cdots. \tag{4.4}$$

Proof. We observe that by (4.1.1) and (4.1.2) Wall polynomials satisfy the recurrence formulae

$$\begin{aligned} B_{n+1}^* &= zB_n^* + \bar\gamma_{n+1}A_n, & A_{n+1}^* &= zA_n^* + \bar\gamma_{n+1}B_n,\\ A_{n+1} &= A_n + \gamma_{n+1}zB_n^*, & B_{n+1} &= B_n + \gamma_{n+1}zA_n^*. \end{aligned} \tag{4.5}$$

Indeed, by Lemma 4.2 we have

$$\begin{aligned} B_{n+1} &= Q_{2n+2} = Q_{2n} + \gamma_{n+1}Q_{2n+1} = B_n + \gamma_{n+1}zA_n^*,\\ B_{n+1}^* &= z^{-1}P_{2n+3} = \bar\gamma_{n+1}P_{2n+2} + (1-|\gamma_{n+1}|^2)\,P_{2n+1}\\ &= |\gamma_{n+1}|^2\,P_{2n+1} + \bar\gamma_{n+1}P_{2n} + (1-|\gamma_{n+1}|^2)\,P_{2n+1}\\ &= zB_n^* + \bar\gamma_{n+1}A_n. \end{aligned}$$

The formulae for A_{n+1}, A_{n+1}^* in (4.5) are proved similarly.

To prove (4.3) we should obtain the recurrence formulae for B_n and A_n separately. By (4.5) we have for $n=2, 3, ...,$

$$\left\{\begin{aligned} B_n &= B_{n-1} + \gamma_n zA_{n-1}^* \\ A_{n-1}^* &= zA_{n-2}^* + \bar\gamma_{n-1}B_{n-2} \\ B_{n-1} &= B_{n-2} + \gamma_{n-1}zA_{n-2}^* \end{aligned}\right. \quad \left|\begin{aligned} &\times\gamma_{n-1}\\ &\times\gamma_n\gamma_{n-1}z\\ &\times(-\gamma_n z). \end{aligned}\right. \tag{4.6}$$

The multipliers in (4.6) are chosen so that all terms A^* are cancelled when we take the sum of the linear equations (4.6),

$$\gamma_{n-1} B_n - \gamma_n z B_{n-1} = \gamma_{n-1} B_{n-1} + |\gamma_{n-1}|^2 \gamma_n z B_{n-2} - \gamma_n z B_{n-2},$$

which implies the required recurrence formula

$$B_n = (1 + \gamma_n z/\gamma_{n-1})\, B_{n-1} - (1 - |\gamma_{n-1}|^2)(\gamma_n z/\gamma_{n-1})\, B_{n-2}. \tag{4.7}$$

One can show similarly that the polynomials A_n also satisfy (4.7):

$$A_n = (1 + \gamma_n z/\gamma_{n-1})\, A_{n-1} - (1 - |\gamma_{n-1}|^2)(\gamma_n z/\gamma_{n-1})\, A_{n-2}. \tag{4.7.1}$$

Since $A_0 = \gamma_0$, $B_0 = 1$ we obtain (4.3).

To prove (4.4) we exclude all terms A from the system

$$\begin{cases} B_n^* = z B_{n-1}^* + \bar{\gamma}_n A_{n-1} & \Big| \times \bar{\gamma}_{n-1} \\ A_{n-1} = A_{n-2} + \gamma_{n-1} z B_{n-2}^* & \Big| \times \bar{\gamma}_n \bar{\gamma}_{n-1} \\ B_{n-1}^* = z B_{n-2}^* + \bar{\gamma}_{n-1} A_{n-2} & \Big| \times (-\bar{\gamma}_n), \end{cases} \tag{4.8}$$

which yields

$$B_n^* = (\bar{\gamma}_n/\bar{\gamma}_{n-1} + z)\, B_{n-1}^* - (1 - |\gamma_{n-1}|^2)(\bar{\gamma}_n z/\bar{\gamma}_{n-1})\, B_{n-2}^*. \tag{4.9}$$

Similarly

$$A_n^* = (\bar{\gamma}_n/\bar{\gamma}_{n-1} + z)\, A_{n-1}^* - (1 - |\gamma_{n-1}|^2)(\bar{\gamma}_n z/\bar{\gamma}_{n-1})\, A_{n-2}^*. \tag{4.9.1}$$

Notice that (4.9) can be obtained from (4.7) by application of the $*$-operation. To complete the proof of (4.4) we observe that $zB_0^* = q_1 \cdot 0 + p_1 \cdot 1$ implies $p_1 = z$ and $zA_0^* = q_1 \cdot 1 + p_1 \cdot 0$ implies $q_1 = \bar{\gamma}_0 z$. Similarly, it follows from $zB_1^* = q_2 \cdot zB_0^* + 0$ that $q_2 = \gamma_0 \bar{\gamma}_1 + z$, while $zA_1^* = q_2 z A_0^* + p_2$ implies $p_2 = z(A_1^* - q_2 \bar{\gamma}_0) = z(\bar{\gamma}_1 + \bar{\gamma}_0 z - |\gamma_0|^2 \bar{\gamma}_1 - \bar{\gamma}_0 z) = (1 - |\gamma_0|^2)\, \bar{\gamma}_1 z$. ∎

The following corollary is immediate from (4.7) and (4.7.1).

COROLLARY 4.3. *For $n = 1, 2, \ldots$*

$$A_n = \gamma_0 + \cdots + \gamma_n z^n, \qquad B_n = 1 + \cdots + \gamma_n \bar{\gamma}_0 z^n. \tag{4.10}$$

COROLLARY 4.4. *For $n = 1, 2, \ldots$*

$$A_n = \gamma_0 + \left\{ \gamma_1 + \gamma_0 \cdot \sum_{\kappa=1}^{n-1} \bar{\gamma}_\kappa \gamma_{\kappa+1} \right\} z + \cdots + \gamma_n z^n \tag{4.11.1}$$

$$B_n = 1 + \left\{ \sum_{\kappa=0}^{n-1} \bar{\gamma}_\kappa \gamma_{\kappa+1} \right\} z + \cdots + \gamma_n \bar{\gamma}_0 z^n \tag{4.11.2}$$

$$A_n^* = \bar{\gamma}_n + \left\{ \bar{\gamma}_{n-1} + \bar{\gamma}_n \sum_{\kappa=0}^{n-2} \bar{\gamma}_\kappa \gamma_{\kappa+1} \right\} z + \cdots + \bar{\gamma}_0 z^n \tag{4.11.3}$$

$$B_n^* = \gamma_0 \bar{\gamma}_n + \left\{ \bar{\gamma}_n \gamma_1 + \bar{\gamma}_{n-1} \gamma_0 + \bar{\gamma}_0 \gamma_0 \cdot \sum_{\kappa=1}^{n-2} \bar{\gamma}_\kappa \gamma_{\kappa+1} \right\} z + \cdots + z^n. \tag{4.11.4}$$

Proof. It follows by induction from (4.5) and Corollary 4.3. ∎

Formulae (4.11.1–4.11.4) are useful for the control of the zeros of the corresponding polynomials.

Notice that (4.5) can be used as a definition of Wall polynomials. With such a definition in mind one can exclude from the consideration the corresponding continued fractions.

The recurrence formulae show that Wall polynomials A_n, B_n are uniquely determined by the parameters $\gamma_0, ..., \gamma_n$. This can also be seen from the formula

$$\begin{aligned} \begin{pmatrix} zB_n^* & -A_n^* \\ -zA_n & B_n \end{pmatrix} &= \prod_{\kappa=0}^{n} \begin{pmatrix} z & -\bar{\gamma}_\kappa \\ -\gamma_\kappa z & 1 \end{pmatrix} \\ &\stackrel{\text{def}}{=} \begin{pmatrix} z & -\bar{\gamma}_n \\ -\gamma_n z & 1 \end{pmatrix} \cdots \begin{pmatrix} z & -\bar{\gamma}_0 \\ -\gamma_0 z & 1 \end{pmatrix}, \end{aligned} \tag{4.12}$$

which is an analogue of (3.9). To obtain (4.12) one should put (4.5) into matrix form and iterate. A similar formula can be found in [3, Sect. 1; 43, Sect. 1]:

$$\begin{pmatrix} A_n^* & B_n^* \\ -B_n & -A_n \end{pmatrix} = \prod_{\kappa=1}^{n} \begin{pmatrix} z & -\bar{\gamma}_\kappa \\ -\gamma_\kappa z & 1 \end{pmatrix} \cdot \begin{pmatrix} \bar{\gamma}_0 & 1 \\ 1 & -\gamma_0 \end{pmatrix}.$$

If we restrict (4.5) to the unit circle, then it is easy to check that

$$\begin{pmatrix} \bar{B}_n & \bar{A}_n \\ A_n & B_n \end{pmatrix} = \prod_{\kappa=0}^{n} \begin{pmatrix} 1 & \bar{z}^\kappa \bar{\gamma}_\kappa \\ z^\kappa \gamma_\kappa & 1 \end{pmatrix}. \tag{4.13}$$

In this form the recurrence formulae (4.5) appeared in [1, (13)].

Basic analytic properties of Wall polynomials follow from *the determinant identity*

$$B_n^* B_n - A_n^* A_n = z^n \prod_{\kappa=0}^{n} (1 - |\gamma_\kappa|^2) \stackrel{\text{def}}{=} \omega_n z^n, \tag{4.14}$$

which is obtained by application of the multiplicative functional $C\mapsto \det(C)$ to both sides of any of the above matrix identities, say (4.12). Restricting (4.14) to the unit circle, we obtain that

$$|B_n(\zeta)|^2-|A_n(\zeta)|^2\equiv\omega_n,\qquad \zeta\in\mathbb{T}. \tag{4.15}$$

LEMMA 4.5 (See [3]). *For n, $n=0, 1, ...,$ the Wall polynomial B_n does not vanish in $\{z: |z|\leqslant 1\}$ and A_n/B_n, $A_n^*/B_n\in\mathscr{B}$.*

Proof. For $n=0$ we have $B_0\equiv 1$, $A_0=\gamma_0$. Suppose now that B_n does not vanish in $\{z: |z|\leqslant 1\}$. Then both functions A_n/B_n, A_n^*/B_n are holomorphic on $\{z: |z|\leqslant 1\}$ and belong to $\mathscr{B}$ by the maximum principle (see (4.15)). By (4.5) we have

$$|B_{n+1}(z)|=|B_n(z)+\gamma_{n+1}zA_n^*(z)|\geqslant|B_n(z)|\,(1-|\gamma_{n+1}|\cdot|A_n^*/B_n|)>0$$

for every z, $|z|\leqslant 1$. ∎

It is clear from (4.15) that $\|A_n/B_n\|_\infty=\|A_n^*/B_n\|_\infty<1$. In fact

$$\|A_n/B_n\|_\infty=\left(1-\frac{\omega_n}{\|B_n\|_\infty^2}\right)^{1/2} \tag{4.16}$$

Since A_n/B_n, $A_n^*/B_n\in\mathscr{B}$, it is natural to compute the Schur parameters of these rational functions. By Theorem 4.2 A_n/B_n is the $(2n)$th approximant of (1.5). Therefore, the Schur parameters of A_n/B_n are given by

$$\gamma_0, \gamma_1, ..., \gamma_n, 0, 0, ... \tag{4.17}$$

This shows that in Schur's theory the approximants A_n/B_n are similar to the Taylor polynomials in the theory of Taylor series.

For A_n^*/B_n we have $Q_{2n+1}/Q_{2n}=zA_n^*/B_n$. By (3.8) we may suppose that the Schur parameters of A_n^*/B_n is the "reversed" sequence $\bar{\gamma}_n, ..., \bar{\gamma}_0$. However, this can be shown directly. By (4.5) we have

$$\frac{A_n^*}{B_n}=\frac{zA_{n-1}^*+\bar{\gamma}_nB_{n-1}}{B_{n-1}+\gamma_nzA_{n-1}^*}=\frac{z(A_{n-1}^*/B_{n-1})+\bar{\gamma}_n}{1+\gamma_nz(A_{n-1}^*/B_{n-1})},$$

which implies that

$$\bar{\gamma}_n, \bar{\gamma}_{n-1}, ..., \bar{\gamma}_0, 0, 0, ... \tag{4.18}$$

is the sequence of the Schur parameters of A_n^*/B_n by (1.3).

THEOREM 4.6 [43, Sect. 1]. *Let A_n, B_n be the Wall polynomials corresponding to a given function f in $\mathscr{B}$ with Schur functions $(f_n)_{n\geqslant 0}$. Then*

$$f(z)=\frac{A_n(z)+zB_n^*(z)\,f_{n+1}(z)}{B_n(z)+zA_n^*(z)\,f_{n+1}(z)}. \tag{4.19}$$

Proof. We apply Lemma 4.1 and Theorem 3.1 to the Wall continued fraction (1.5). By (1.4), we obtain

$$f=\tau_0\circ\tau_1\circ\cdots\circ\tau_n(f_{n+1})=s_0\circ s_1\circ\cdots\circ s_{2n+1}(1/f_{n+1})$$
$$=S_{2n+1}(1/f_{n+1})=\frac{P_{2n}+P_{2n+1}\,f_{n+1}}{Q_{2n}+Q_{2n+1}\,f_{n+1}}=\frac{A_n+zB_n^*\,f_{n+1}}{B_n+zA_n^*\,f_{n+1}}. \quad ∎$$

Wall polynomials provide a simple description of the set $\mathscr{E}_n=\mathscr{E}_n(f)$ consisting of all functions in $\mathscr{B}$ with a fixed set of first Schur parameters $\{\gamma_0, \gamma_1, ..., \gamma_n\}$:

$$\mathscr{E}_n=\left\{\frac{A_n+zB_n^*\,\mathscr{E}}{B_n+zA_n^*\,\mathscr{E}}:\mathscr{E}\in\mathscr{B}\right\}. \tag{4.20}$$

This follows from (1.4) by Theorem 4.6. By the determinant identity (4.14) we have for every $\mathscr{E}$ in $\mathscr{B}$ that

$$\frac{A_n+zB_n^*\,\mathscr{E}}{B_n+zA_n^*\,\mathscr{E}}-\frac{A_n}{B_n}=z^{n+1}\mathscr{E}\,\frac{\omega_n}{B_n(B_n+zA_n^*\,\mathscr{E})}, \tag{4.21}$$

which implies that all functions in $\mathscr{E}_n$ have the same Taylor polynomial of degree n at $z=0$ [12, Chap. IV, Exercise 21].

COROLLARY 4.7 [54, Theorem A]. *Let $f\in\mathscr{B}$ and let A_n, B_n be the Wall polynomials associated with f. Then*

$$\frac{A_n}{B_n}\rightrightarrows f$$

uniformly on compact subsets of $\mathbb{D}$.

Proof. We put $\mathscr{E}=f_{n+1}$ in (4.21) and by Theorem 4.6 obtain

$$f-\frac{A_n}{B_n}=z^{n+1}f_{n+1}\,\frac{\omega_n}{B_n^2(1+zf_{n+1}\cdot A_n^*/B_n)}. \tag{4.22}$$

By Lemma 4.5 and (4.15) we conclude that $\omega_n \cdot B_n^{-2} \in \mathcal{B}$. Since $A_n^*/B_n \in \mathcal{B}$ by Lemma 4.5, we obtain that

$$\left| f(z) - \frac{A_n}{B_n}(z) \right| \leqslant \frac{|z|^{n+1}}{(1-|z|)}, \qquad z \in \mathbb{D},$$

which completes the proof. ∎

The following lemma is useful for the study of the pointwise convergence of A_n/B_n on $\mathbb{T}$.

LEMMA 4.8. (1) *Let $\zeta \in \mathbb{T}$ and let $|f(\zeta)| < 1$. Then*

$$\lim_n \frac{A_n}{B_n}(\zeta) = f(\zeta) \tag{4.23}$$

if and only if $\lim_n |f_n(\zeta)| = 0$.

(2) *Let $F = \{\zeta \in \mathbb{T} : |f(\zeta)| = 1\}$ and let $mF > 0$. Then $A_n/B_n \rightrightarrows f$ on F if and only if $|A_n/B_n| \rightrightarrows 1$ on F.*

Proof. (1) If $\lim_n |f_n(\zeta)| = 0$, then (4.23) holds by (4.22). If (4.23) holds and $|f(\zeta)| < 1$, then $|A_n^*(\zeta)/B_n(\zeta)| \to |f(\zeta)| < 1$. By (4.15) $\omega_n \cdot |B_n(\zeta)|^{-2} \to 1 - |f(\zeta)|^2 > 0$. It follows from (4.22), that $\lim_n f_n(\zeta) = 0$.

(2) By Cauchy's inequality (see (4.15) and (4.22))

$$\begin{aligned}\int_F \left| f - \frac{A_n}{B_n} \right|^p dm &= \int_F \left(1 - \left| \frac{A_n}{B_n} \right|^2 \right)^p \frac{dm}{|1 + z f_{n+1} \cdot A_n^*/B_n|^p} \\ &\leqslant \left(\int_F \left(1 - \left| \frac{A_n}{B_n} \right|^2 \right)^{2p} dm \right)^{1/2} \\ &\quad \times \left(\int_F \frac{dm}{|1 + z f_{n+1} \cdot A_n^*/B_n|^{2p}} \right)^{1/2}. \end{aligned} \tag{4.24}$$

If $2p < 1$, then the second integral on the right-hand side of (4.24) is uniformly bounded by Smirnov's theorem [12, Chap. III, Theorem 2.4]. It follows that $A_n/B_n \to f$ in the L^p-metric on F by Lebesgue's dominated convergence theorem [53, Chap. VII, Sect. 3, Theorem VIII.3.1] if we assume that $|A_n/B_n| \rightrightarrows 1$ on F. The converse conclusion is obvious. ∎

Returning to formula (4.19), we summarize a number of useful identities for Wall polynomials in the following theorem.

THEOREM 4.9. *Let $f \in \mathscr{B}$. Then*

$$B_n + A_n^* z f_{n+1} = \prod_{\kappa=0}^{n} (1 + z\bar{\gamma}_\kappa f_{\kappa+1}), \tag{4.25.1}$$

$$A_n + B_n^* z f_{n+1} = (\gamma_0 + z f_1) \prod_{\kappa=1}^{n} (1 + z\bar{\gamma}_\kappa f_{\kappa+1}), \tag{4.25.2}$$

$$\begin{aligned} B_n f - A_n &= z^{n+1}\omega_n f_{n+1} \prod_{\kappa=0}^{n} (1 + z\bar{\gamma}_\kappa f_{\kappa+1})^{-1} \\ &= z^{n+1} f_{n+1} \cdot \prod_{\kappa=0}^{n} (1 - \bar{\gamma}_\kappa f_\kappa), \end{aligned} \tag{4.25.3}$$

$$\begin{aligned} B_n^* - A_n^* f &= z^n \omega_n \prod_{\kappa=0}^{n} (1 - z\bar{\gamma}_\kappa f_{\kappa+1})^{-1} \\ &= z^n \cdot \prod_{\kappa=0}^{n} (1 - \bar{\gamma}_\kappa f_\kappa). \end{aligned} \tag{4.25.4}$$

Proof. The first identity can be obtained by Lemma 3.3 or can be proved by induction [43, p. 295]. We obtain (4.25.1) by induction using the useful identity

$$1 - |\gamma_\kappa|^2 = (1 - \bar{\gamma}_\kappa f_\kappa)(1 + \bar{\gamma}_\kappa z f_{\kappa+1}). \tag{4.26}$$

Assuming that (4.25.1) holds for indices smaller than n, we obtain by (4.5), (1.3), and (4.26) that

$$\begin{aligned} B_n + A_n^* z f_{n+1} &= (B_{n-1} + \gamma_n z A_{n-1}^*) + (z A_{n-1}^* + \bar{\gamma}_n B_{n-1}) \cdot \frac{f_n - \gamma_n}{1 - \bar{\gamma}_n f_n} \\ &= \frac{(B_{n-1} + z A_{n-1}^* f_n)(1 - |\gamma_n|^2)}{1 - \bar{\gamma}_n f_n} \\ &= (B_{n-1} + z A_{n-1}^* f_n)(1 + \bar{\gamma}_n z f_{n+1}). \end{aligned}$$

The second identity follows from (4.25.1) by Theorem 4.6.

To obtain (4.25.3) we multiply (4.25.2) by B_n and subtract (4.25.1) multiplied by A_n from the resulting identity. Now (4.25.3) follows by (4.14) and (4.26).

The identity (4.25.4) is proved similarly. ∎

COROLLARY 4.10. *for $f \in \mathscr{B}$ we have*

$$B_n = \left\{ \frac{1}{1} + \frac{\bar{\gamma}_n z f_{n+1}}{1} - \frac{(1-|\gamma_n|^2)(\bar{\gamma}_{n-1}/\bar{\gamma}_n)\, z}{1+(\bar{\gamma}_{n-1}/\bar{\gamma}_n)\, z} - \cdots - \frac{(1-|\gamma_1|^2)(\bar{\gamma}_0/\bar{\gamma}_1)\, z}{1+(\bar{\gamma}_0/\bar{\gamma}_1)\, z} \right\} \cdot \prod_{\kappa=0}^{n} (1 + z\bar{\gamma}_\kappa f_{\kappa+1}), \tag{4.27.1}$$

$$A_n = f \cdot B_n - z^{n+1} f_{n+1} \cdot \prod_{\kappa=0}^{n} (1 - \bar{\gamma}_\kappa f_\kappa). \tag{4.27.2}$$

Proof. Since the Schur parameters of A_n^*/B_n are given by (4.18), we obtain by Theorem 4.2 that

$$\frac{A_n^*}{B_n} = \frac{\bar{\gamma}_n}{1} - \frac{(1-|\gamma_n|^2)(\bar{\gamma}_{n-1}/\bar{\gamma}_n)\, z}{1+(\bar{\gamma}_{n-1}/\bar{\gamma}_n)\, z} - \cdots - \frac{(1-|\gamma_1|^2)(\bar{\gamma}_0/\bar{\gamma}_1)\, z}{1+(\bar{\gamma}_0/\bar{\gamma}_1)\, z}, \tag{4.28}$$

which implies (4.27.1) by (4.25.1). Clearly, (4.27.2) is equivalent to (4.25.3). ▮

Notice that by comparing the terms in z in (4.25.1–4.25.4) one can easily obtain (4.11.1–4.11.4).

The following lemma is a convenient tool in the study of Schur functions.

LEMMA 4.11. *Let $(f^\kappa)_{\kappa \geqslant 0}$ be a sequence of functions in $\mathscr{B}$, $(\gamma_n^\kappa)_{n \geqslant 0}$ Schur parameters of f^κ, and $(f_n^\kappa)_{n \geqslant 0}$ the Schur functions of f^κ. Suppose that*

$$\lim_\kappa f^\kappa(z) = f(z) \tag{4.29}$$

uniformly on compact subsets of $\mathbb{D}$. Let $(\gamma_n)_{n \geqslant 0}$ be the Schur parameters of f, and let $(f_n)_{n \geqslant 0}$ be the Schur functions of f. Then for every n

$$\lim_\kappa f_n^\kappa(z) = f_n(z) \tag{4.30}$$

uniformly on compact subsets of $\mathbb{D}$ and, in particular,

$$\lim_\kappa \gamma_n^\kappa = \gamma_n. \tag{4.31}$$

Proof. For $n = 0$ (4.29) and (4.30) are equivalent. We have

$$z f_{n+1}^\kappa = \frac{f_n^\kappa - \gamma_n^\kappa}{1 - \bar{\gamma}_n^\kappa f_n^\kappa}. \tag{4.32}$$

If (4.30) holds for n, then $\lim_\kappa \gamma_n^\kappa = \gamma_n$ (put $z=0$ in (4.30)). If $|\gamma_n|=1$, then there is nothing to prove, since $f_n \equiv \gamma_n$ and f is a finite Blaschke product of order n. So f_{n+1} does not exist. If $|\gamma_n|<1$, then we have

$$\frac{f_n^\kappa - \gamma_n^\kappa}{1-\bar{\gamma}_n^\kappa f_n^\kappa} - \frac{f_n - \gamma_n}{1-\bar{\gamma}_n f_n}$$
$$= \frac{(f_n^\kappa - f_n) + (\gamma_n - \gamma_n^\kappa) + (\bar{\gamma}_n^\kappa - \bar{\gamma}_n) f_n f_n^\kappa + \gamma_n^\kappa \bar{\gamma}_n f_n - \gamma_n \bar{\gamma}_n^\kappa f_n^\kappa}{(1-\bar{\gamma}_n^\kappa f_n^\kappa)(1-\bar{\gamma}_n f_n)}.$$

It follows that for any compact subset F of $\mathbb{D}$, $0 \in F$, we have

$$\sup_F \left| \frac{f_n^\kappa - \gamma_n^\kappa}{1-\bar{\gamma}_n^\kappa f_n^\kappa} - \frac{f_n - \gamma_n}{1-\bar{\gamma}_n f_n} \right| \leqslant \frac{6 \sup_F |f_n^\kappa - f_n|}{(1-|\gamma_n^\kappa|)(1-|\gamma_n|)},$$

which obviously implies that $\sup_F |f_{n+1}^\kappa - f_{n+1}| \to 0$ as $\kappa \to +\infty$. ∎

It follows from Corollary 4.7 (Wall's theorem) that the Schur parameters uniquely determine the corresponding function f in $\mathscr{B}$. This, together with compactness of $\mathscr{B}$ in the topology of uniform convergence, implies that the converse to Lemma 4.11 is also true. Indeed, suppose that (4.31) holds for every n and let g be any limit point of $(f^\kappa)_{\kappa \geqslant 0}$. Applying Lemma 4.11 to a subsequence of $(f^\kappa)_{\kappa \geqslant 0}$, we obtain that $g=f$, since g and f have identical Schur's parameters.

Corollary 4.12. *Let $f \in \mathscr{B}$. Then $\lim_n \gamma_n = 0$ if and only if*

$$f_n(z) \rightrightarrows 0$$

uniformly on compact subsets of $\mathbb{D}$.

Proof. Obviously the sequence $(\gamma_{\kappa+n})_{n\geqslant 0}$ is the sequence of the Schur parameters of f_κ. Now we apply the converse of Lemma 4.11 to f^κ, $f^\kappa \stackrel{\text{def}}{=} f_\kappa$, $\kappa = 0, 1, \ldots$. ∎

The following theorem provides two useful representations for Schur functions.

Theorem 4.13. *Let $f \in \mathscr{B}$, $(f_n)_{n\geqslant 0}$ be the Schur functions of f, and $(\gamma_n)_{n\geqslant 0}$ the Schur parameters of f. Let $(A_n)_{n\geqslant 0}$, $(B_n)_{n\geqslant 0}$ be the Wall polynomials corresponding to the parameters $(\gamma_n)_{n\geqslant 0}$. Then*

$$f(z)=\sum_{n=0}^{\infty}\gamma_n z^n\cdot\prod_{\kappa=0}^{n-1}(1-\bar{\gamma}_\kappa f_\kappa), \tag{4.33.1}$$

$$f(z)=\gamma_0+\sum_{n=0}^{\infty}\gamma_{n+1}z^{n+1}\frac{\omega_n}{B_n+zA_n^* f_{n+1}}, \tag{4.33.2}$$

where both series converge uniformly on compact subsets of $\mathbb{D}$.

Proof. Iterating an obvious identity

$$f_n(z)=\gamma_n+(1-\bar{\gamma}_n f_n)\,zf_{n+1},$$

we obtain

$$\begin{aligned} f(z)=\gamma_0&+(1-\bar{\gamma}_0 f_0)\,\gamma_1 z+(1-\bar{\gamma}_0 f_0)(1-\bar{\gamma}_1 f_1)\,\gamma_2 z^2+\cdots\\ &+(1-\bar{\gamma}_0 f_0)\cdots(1-\bar{\gamma}_{n-1} f_{n-1})\,z^n\cdot f_n. \end{aligned} \tag{4.34}$$

By (4.25.1) and (4.26) we have

$$\frac{\omega_n}{B_n+zA_n^* f_{n+1}}=\prod_{\kappa=0}^{n}(1-\bar{\gamma}_\kappa f_\kappa). \tag{4.35}$$

Finally,

$$\left|\frac{\omega_n}{B_n+zA_n^* f_{n+1}}\right|=\frac{\sqrt{\omega_n}}{|B_n|}\cdot\frac{\sqrt{\omega_n}}{|1+zf_{n+1}A_n^*/B_n|}\leqslant\frac{\sqrt{\omega_n}}{1-|z|} \tag{4.36}$$

completes the proof. ∎

Notice that by (4.36) the convergence of (4.33.1) and (4.33.2) on any compact subset of $\mathbb{D}$ is uniform on the ball $\mathscr{B}$. Applying (4.33.2) to the family $(f_n)_{n\geqslant 0}$, we obtain another proof of Corollary 4.12. One can prove similarly Lemma 4.11, as well as its converse.

A representation similar to (4.33.2) can be obtained by Wall's theorem. By (4.5) we have

$$\frac{A_{n+1}}{B_{n+1}}-\frac{A_n}{B_n}=\gamma_{n+1}z^{n+1}\frac{\omega_n}{B_nB_{n+1}},\qquad n=0,1,..., \tag{4.37}$$

which obviously implies

$$f(z)=\gamma_0+\sum_{n=0}^{\infty}\gamma_{n+1}z^{n+1}\frac{\omega_n}{B_nB_{n+1}}. \tag{4.38}$$

Since $\omega_n \cdot B_n^{-2} \in \mathscr{B}$, we have (compare with (4.36))

$$\left|\frac{\omega_n}{B_n B_{n+1}}\right| = \frac{\omega_n}{|B_n|^2} \cdot \frac{1}{|1+z\gamma_{n+1}A_n^*/B_n|} \leqslant \frac{1}{1-|z|}. \tag{4.39}$$

However, there is an essential difference in the behavior of (4.33.2) and (4.38) on the unit circle $\mathbb{T}$.

THEOREM 4.14. *For every* $f \in \mathscr{B}$ *the series* (4.33.2) *converges to* f *in* $L^p(\mathbb{T})$ *for every* p, $0<p<1$.

Proof. Given $p<1$ we take any r, $r>1$, with $rp<1$ and obtain by Hölder's inequality

$$\begin{aligned}
&\int_{\mathbb{T}} \frac{\omega_n^p}{|B_n + zA_n^* f_{n+1}|^p} \cdot |f_{n+1}|^p \, dm \\
&\quad \leqslant \left(\int_{\mathbb{T}} \frac{\omega_n^{rp}}{|B_n + zA_n^* f_{n+1}|^{rp}} \, dm\right)^{1/r} \left(\int_{\mathbb{T}} |f_{n+1}|^{r'p} \, dm\right)^{1/r'} \\
&\quad \leqslant \omega_n^{p/2} \cdot \left(\int_{\mathbb{T}} |f_{n+1}|^{r'p} \, dm\right)^{1/r'} \left(\int_{\mathbb{T}} \frac{dm}{|1+zf_{n+1}A_n^*/B_n|^{rp}}\right)^{1/r} \\
&\quad \leqslant C_p \cdot \omega_n^{p/2} \left(\int_{\mathbb{T}} |f_{n+1}|^{r'p} \, dm\right)^{1/r'},
\end{aligned} \tag{4.40}$$

by Smirnov's theorem [12, Chap. III, Sect. 2, Theorem 2.4] since $\operatorname{Re}(1+zf_{n+1}A_n^*/B_n)>0$ and $\sqrt{\omega_n} \cdot |B_n|^{-1} \leqslant 1$, see (4.15). Now if f is the Schur function of a Szegő measure σ, then the integral in the left-hand side of (4.40) tends to zero, since by (2.11) $f_{n+1} \Rightarrow 0$ on $\mathbb{T}$ and since ω_n, $\|f_{n+1}\|_\infty \leqslant 1$. If σ is not a Szegő measure, then by (5.2) $\lim_n \omega_n = 0$. It follows that for any $f \in \mathscr{B}$

$$\lim_n \int_{\mathbb{T}} \frac{\omega_n^p}{|B_n + zA_n^* f_{n+1}|^p} \cdot |f_{n+1}|^p \, dm = 0,$$

which proves the theorem by (4.34) and (4.35). ▮

To the contrary, the remaining term in (4.38) is obviously $f - A_n/B_n$, which by Theorem 5 tends to zero in measure not for every f in $\mathscr{B}$.

There is a natural question on the convergence properties of the odd approximants $P_{2n+1}/Q_{2n+1} = B_n^*/A_n^*$, see (4.2), of (1.5) in $\mathbb{D}$.

THEOREM 4.15. *Suppose that the Schur parameters $(\gamma_n)_{n \geqslant 0}$ of f satisfy*

$$\sum_{n=0}^{\infty} |\gamma_n|^2 < +\infty.$$

Then $\lim_n B_n^*/A_n^*$ *exists at* z, $z \in \mathbb{D}$, *if and only if*

$$\lim_n \frac{z^n}{A_n^*(z)} = O.$$

Proof. It is immediate from (4.25.4) by Theorem 5.15 (see Section 5 below). ∎

Theorem 4.15 is due to Njåstad [42]. We see that the convergence of the odd pat of (1.5) is related to the distribution of the zeros of A_n^* in $\mathbb{D}$. Similarly, the even part of the Möbius transform $w \mapsto (1+zw)(1-zw)^{-1}$ of (1.5) regulates the distribution of the zeros of the orthogonal polynomial (see (5.5) below).

5. ORTHOGONAL POLYNOMIALS

Let $(\varphi_n)_{n \geqslant 0}$ be orthogonal polynomials in $L^2(d\sigma)$ and $(a_n)_{n \geqslant 0}$ be the Geronimus parameters of σ (see Section 1). Since obviously $\varphi_n^* \perp z, z^2, ..., z^n$, $n = 1, ...,$ in $L^2(d\sigma)$, we obtain the following formula [15]

$$k_n \cdot \varphi_n^*(z) = \sum_{j=0}^{n} \overline{\varphi_j(0)} \cdot \varphi_j(z), \tag{5.1}$$

which implies (put $z = 0$) that

$$1 - |a_n|^2 = 1 - \frac{|\varphi_{n+1}(0)|^2}{k_{n+1}^2} = \left(\frac{k_n}{k_{n+1}}\right)^2,$$

and consequently that

$$k_{n+1}^{-2} = \prod_{j=0}^{n} (1 - |a_j|^2). \tag{5.2}$$

It follows that the orthogonal polynomials $(\psi_n)_{n \geqslant 0}$ with the parameters $(-a_n)_{n \geqslant 0}$ have the same leading coefficients $(k_n)_{n \geqslant 0}$.

The recurrence formulae (1.11) for the polynomials $(\varphi_n)_{n\geqslant 0}$ and $(\psi_n)_{n\geqslant 0}$ can be put into the matrix form

$$\begin{pmatrix} \varphi_{n+1} & \psi_{n+1} \\ \varphi^*_{n+1} & -\psi^*_{n+1} \end{pmatrix} = \frac{k_{n+1}}{k_n} \begin{pmatrix} z & -\bar{a}_n \\ -a_n z & 1 \end{pmatrix} \begin{pmatrix} \varphi_n & \psi_n \\ \varphi^*_n & -\psi^*_n \end{pmatrix} \tag{5.3}$$

(cf. [43, (15)]). Iterating (5.3), we obtain an explicit formula for orthogonal polynomials

$$\begin{pmatrix} \varphi_{n+1} & \psi_{n+1} \\ \varphi^*_{n+1} & -\psi^*_{n+1} \end{pmatrix} = k_{n+1} \cdot \prod_{\kappa=0}^{n} \begin{pmatrix} z & -\bar{a}_\kappa \\ -a_\kappa z & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{5.4}$$

The matrix product in (5.4) equals the matrix (4.12) of the Wall polynomials corresponding to the Schur parameters $(a_n)_{n\geqslant 0}$. Hence we obtain simple formulae relating orthogonal polynomials with Wall polynomials (cf. [43, Theorem 5]):

$$\begin{aligned} \varphi_{n+1} &= k_{n+1}(zB^*_n - A^*_n) \qquad & \psi_{n+1} &= k_{n+1}(zB^*_n + A^*_n) \\ \varphi^*_{n+1} &= k_{n+1}(B_n - zA_n) \qquad & \psi^*_{n+1} &= k_{n+1}(B_n + zA_n). \end{aligned} \tag{5.5}$$

By (5.5) and by Lemma 4.5 the polynomials φ^*_n, ψ^*_n do not vanish in $\{z: |z| \leqslant 1\}$. We define by

$$\Phi_n(z) = k_n^{-1} \varphi_n(z), \qquad \Psi_n(z) = k_n^{-1} \cdot \psi_n(z) \tag{5.6}$$

the monic orthogonal polynomials. The following theorem shows that $(\Psi^*_n)_{n\geqslant 0}$ are the numerators and $(\Phi^*_n)_{n\geqslant 0}$ are the denominators of a continued fraction.

THEOREM 5.1 [13, Theorem 5.2]. *Let σ be a probability measure on $\mathbb{T}$, $(\varphi_n)_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$, $(a_n)_{n\geqslant 0}$ the Geronimus parameters of σ, and $(\psi_n)_{n\geqslant 0}$ the orthogonal polynomials associated with the parameters $(-a_n)_{n\geqslant 0}$. Then the sequence $1/0$, Ψ^*_0/Φ^*_0, ..., Ψ^*_n/Φ^*_n, ... is the sequence of the approximants of the continued fraction*

$$\begin{aligned} C = 1 + \frac{2a_0 z}{1 - a_0 z} - \frac{(1-|a_0|^2)(a_1/a_0)\, z}{1 + (a_1/a_0)\, z} - \cdots \\ - \frac{(1-|a_{n-2}|^2)(a_{n-1}/a_{n-2})\, z}{1 + (a_{n-1}/a_{n-2})\, z} - \cdots. \end{aligned} \tag{5.7}$$

Proof. By (5.5), (4.7), and (4.7.1) the sequences $(\Phi_n^*)_{n\geqslant 2}$ and $(\Psi_n^*)_{n\geqslant 2}$ satisfy the recurrence equation

$$y_n = q_n y_{n-1} + p_n y_{n-2}$$

with $q_n = 1 + a_{n-1}z/a_{n-2}$, $p_n = -(1-|a_{n-2}|^2)(a_{n-2}z/a_{n-1})$, $n = 2, 3, \dots$. If we put $\Phi^*_{-1} = 0$, $\Psi^*_{-1} = 1$, then we obtain by (5.5) that

$$\Phi_1^* = -a_0 z + 1 = (1-a_0 z)\,\Phi_0^* + 2a_0 z \Phi^*_{-1},$$
$$\Psi_1^* = 1 + a_0 z = (1-a_0 z)\,\Psi_0^* + 2a_0 z \Psi^*_{-1},$$

which completes the proof. ▮

Applying the multiplicative functional $C \mapsto \det(C)$ to (5.4) and using (5.2) we obtain

$$\varphi_n \psi_n^* + \varphi_n^* \psi_n = 2z^n \tag{5.8}$$

(compare with (4.14)), which implies that

$$\operatorname{Re} \frac{\psi_n^*}{\varphi_n^*} = \frac{1}{2}\left(\frac{\psi_n^*}{\varphi_n^*} + \frac{\bar{\psi}_n^*}{\bar{\varphi}_n^*}\right) = \frac{\bar{z}^n(\varphi_n \psi_n^* + \varphi_n^* \psi_n)}{2\,|\varphi_n|^2} = \frac{1}{|\varphi_n|^2} \tag{5.9}$$

on $\mathbb{T}$. It follows by Schwarz's formula [7, Ch. VIII, Section 3, (3.4)] that

$$\frac{\psi_n^*}{\varphi_n^*} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \frac{dm}{|\varphi_n(\zeta)|^2} \tag{5.10}$$

since $\psi_n^*(0)/\varphi_n^*(0) = k_n/k_n = 1 = \int_{\mathbb{T}} |\varphi_n|^{-2}\,dm$. By (5.5) and (5.10) we have

$$\frac{1 + z(A_n/B_n)}{1 - z(A_n/B_n)} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \frac{dm}{|\varphi_{n+1}(\zeta)|^2}. \tag{5.11}$$

The following corollary is immediate from (5.11).

COROLLARY 5.2. *A_n/B_n is the Schur function of the probability measure $|\varphi_{n+1}|^{-2}\,dm$.*

The following theorem is well known [30]. We provide a simple proof for completeness.

THEOREM 5.3. *Let $(\varphi_n)_{n\geqslant 0}$ be a sequence of polynomials satisfying the recurrence formulae* (1.11). *Then, for every n, the polynomials $\varphi_0, \varphi_1, \dots, \varphi_n$ are orthogonal in $L^2(|\varphi_n|^{-2}\,dm)$.*

Proof. It is sufficient to prove that

$$\int_{\mathbb{T}} \bar{p}\varphi_\kappa \frac{dm}{|\varphi_n|^2}=0, \qquad \int_{\mathbb{T}} \overline{zp}\varphi_n^* \frac{dm}{|\varphi_n|^2}=0$$

for every κ, $\kappa \leqslant n$, and for every polynomial p in $\mathscr{P}_{n-1}$.

If $\kappa = n$, then by the mean-value theorem

$$\int_{\mathbb{T}} \frac{\bar{p}\varphi_n}{\varphi_n \bar{\varphi}_n}\, dm = \int_{\mathbb{T}} \frac{z^n \bar{p}}{\varphi_n^*}\, dm = \int_{\mathscr{T}} \frac{zp^*}{\varphi_n^*}\, dm = 0,$$

and similarly

$$\int_{\mathbb{T}} \frac{\overline{zp}}{\varphi_n^*} \cdot \frac{\varphi_n^*}{\bar{\varphi}_n^*}\, dm = \overline{\int_{\mathbb{T}} \frac{zp}{\varphi_n^*}\, dm} = 0.$$

This implies that $\varphi_n \perp 1,\ z, ..., z^{n-1}$, and that $\varphi_n^* \perp z,\ z^2, ...,\ z^n$ in $L^2(|\varphi_n|^{-2}\, dm)$. Now we present (5.1) and the first formula of (1.11) as follows

$$\begin{aligned} k_n \varphi_n^* &= k_{n-1} \varphi_{n-1}^* + \overline{\varphi_n(0)}\, \varphi_n \\ k_{n-1} \varphi_n &= k_n z \varphi_{n-1} + \varphi_n(0)\, \varphi_{n-1}^*. \end{aligned} \tag{5.12}$$

From the first formula we conclude that $\varphi_{n-1}^* \perp z,\ z^2, ...,\ z^{n-1}$ in $L^2(|\varphi_n|^{-2}\, dm)$, which together with the second formula yields $\varphi_{n-1} \perp 1, ...,$ z^{n-2} in $L^2(|\varphi_n|^{-2}\, dm)$. Clearly, we can continue these arguments with (5.12) by induction. ∎

We observe that *Geronimus' theorem* (see Section 1) is an easy consequence of Theorem 5.3 and Corollary 5.2. Indeed, given a probability measure σ, by Theorem 5.3, we obtain that

$$\int_{\mathbb{T}} \bar{\zeta}^\kappa\, |\varphi_{n+1}(\zeta)|^{-2}\, dm = \int_{\mathbb{T}} \bar{\zeta}^\kappa\, d\sigma \tag{5.13}$$

for $\kappa = 0,\ \pm 1, ...,\ \pm (n+1)$, which implies that

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \frac{dm}{|\varphi_{n+1}|^2} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma + O(z^{n+2}), \qquad z \to 0. \tag{5.14}$$

Let A_n, B_n be the Wall polynomials associated with the Geronimus parameters $a_0, ..., a_n$ of σ (or of $|\varphi_{n+1}|^{-2}\, dm$ by Theorem 5.3). If f is the Schur function of σ, then by Corollary 5.2 and by (5.14) we obtain

$$\frac{A_n}{B_n} = f + O(z^{n+1}),$$

which implies that $A_n/B_n \in \mathscr{E}_n(f)$ (see (4.20)–(4.21)) and therefore $a_n = \gamma_n$. ∎

The following well-known lemma is a cornerstone of the method of weak and strong convergence. It is immediate from (5.13) by Weierstrass' theorem.

Lemma 5.4. *Let σ be a probability measure on $\mathbb{T}$ with infinite support and $(\varphi_n)_{n \geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then*

$$(*)\text{-}\lim_n |\varphi_n|^{-2}\, dm = d\sigma. \tag{5.15}$$

It follows from Lemma 5.4 and (5.10) that the continued fraction (5.7) converges to the Schwarz integral F_σ uniformly on compact subsets of $\mathbb{D}$.

We can now illustrate the method of weak and strong convergence (see Section 2) with a simple proof of Szegő's classical theorem.

Since $f \mapsto zf$ is an isometry and $f \mapsto z^{n-1} \cdot \bar{f}$ is an antilinear isometry in $L^2(d\sigma)$, which maps $\mathscr{P}_{n-1}$ onto itself, we obtain that

$$\operatorname{dist}(z^n, \mathscr{P}_{n-1}) = \operatorname{dist}(\bar{z}, \mathscr{P}_{n-1}) = \operatorname{dist}(\mathbb{1}, z\mathscr{P}_{n-1}) = k_n^{-1}. \tag{5.16}$$

In agreement with (5.10) this shows that $(k_n)_{n \geqslant 0}$ is an increasing sequence.

We say that σ is a *Szegő measure* if $\lim_n k_n = k < +\infty$.

Theorem 5.5. *A probability measure σ is a Szegő measure if and only if $\int_{\mathbb{T}} \log \sigma'\, dm > -\infty$. Moreover, for any probability measure σ on $\mathbb{T}$ we have*

$$\lim_n \frac{1}{k_n^2} = \exp\left(\int_{\mathbb{T}} \log \sigma'\, dm\right). \tag{5.17}$$

Proof. Since φ_n^* does not vanish in $\{z\colon |z| \leqslant 1\}$, the function $\log |\varphi_n^*|^{-2}$ is harmonic on $\{z\colon |z| \leqslant 1\}$ and therefore by the mean-value theorem we obtain that

$$\int_{\mathbb{T}} \log \frac{1}{|\varphi_n|^2}\, dm = \log \frac{1}{|\varphi_n^*(0)|^2} = \log \frac{1}{k_n^2}. \tag{5.18}$$

Suppose that $\int_{\mathbb{T}} \log \sigma' \, dm > -\infty$. Then by Jensen's inequality and by (5.18)

$$\begin{aligned} \int_{\mathbb{T}} \log \sigma' \, dm &= \int_{\mathbb{T}} \log \left(\left| \frac{\varphi_n^*}{k_n} \right|^2 \sigma' \right) dm \\ &\leqslant \log \left(\int_{\mathbb{T}} \left| \frac{\varphi_n^*}{k_n} \right|^2 d\sigma \right) = \log \frac{1}{k_n^2}, \end{aligned} \tag{5.19}$$

which obviously implies that σ is a Szegő measure.

Suppose now that σ is a Szegő measure. In what follows we use the standard notations $u^+ = \max(u, 0)$, $u^- = u^+ - u$.

Observing that $(\log^+ x)^2 \leqslant x$, $x > 0$, and that by (5.10) $|\varphi_n|^{-2} \, dm$ is a probability measure, we conclude that

$$\int_{\mathbb{T}} \left(\log^+ \frac{1}{|\varphi_n|^2} \right)^2 dm \leqslant \int_{\mathbb{T}} |\varphi_n|^{-2} \, dm = 1, \tag{5.20}$$

which by (5.18) implies

$$\int_{\mathbb{T}} \log^- \frac{1}{|\varphi_n|^2} \, dm = \int_{\mathbb{T}} \log^+ \frac{1}{|\varphi_n|^2} \, dm + \log k_n^2 \leqslant 1 + \log k^2. \tag{5.21}$$

Now, consider the sequence $d\mu_n = \log \frac{1}{|\varphi_n|^2} \, dm$, $n = 0, 1, ...,$ of real Borel measures on $\mathbb{T}$. Clearly, $\mu_n = \mu_n^+ - \mu_n^-$, where $d\mu_n^\pm = \log^\pm \frac{1}{|\varphi_n|^2} \, dm$.

By (5.21) the sequence $(\mu_n^-)_{n \geqslant 0}$ is bounded in $M(\mathbb{T})$. Let ν be any $(*)$-limit point of $(\mu_n^-)_{n \geqslant 0}$. Then there exists a set Λ of positive integers such that

$$(*)\text{-}\lim_{n \in \Lambda} d\mu_n^- = \nu' \, dm + d\nu_s, \tag{5.22}$$

where $d\nu_s$ is the singular part of $d\nu$ and ν' is the Lebesgue derivative of ν. Since the unit ball of the Hilbert space $L^2(\mathbb{T})$ is weakly compact, by (5.20) any $(*)$-limit point ω of $(\mu_n^+)_{n \in \Lambda}$ is absolutely continuous with respect to dm. It follows that there exists a subset Λ' of Λ such that

$$\begin{aligned} (*)\text{-}\lim_{n \in \Lambda'} d\mu_n^+ &= \omega' \, dm \\ (*)\text{-}\lim_{n \in \Lambda'} d\mu_n &= d\mu, \end{aligned} \tag{5.23}$$

where

$$d\mu = (\omega' - v')\, dm - dv_s. \tag{5.24}$$

Let I be any open arc on $\mathbb{T}$ such that its end-points do not carry point masses of the singular measures dv_s and $d\sigma_s$. By Jensen's inequality we obtain

$$\exp\left\{\frac{1}{|I|}\int_I \log\frac{1}{|\varphi_n|^2}\, dm\right\} \leqslant \frac{1}{|I|}\int_I \frac{dm}{|\varphi_n|^2}. \tag{5.25}$$

Applying Helly's theorem separately to $(\mu_n^+)_{n\in\Lambda'}$ and to $(\mu_n^-)_{n\in\Lambda'}$, we obtain that

$$\lim_{n\in\Lambda'} \frac{1}{|I|}\int_I \log\frac{1}{|\varphi_n|^2}\, dm = \frac{\mu(I)}{|I|}. \tag{5.26}$$

Applying Lemma 5.4 and Helly's theorem, we obtain

$$\lim_{n} \frac{1}{|I|}\int_I \frac{dm}{|\varphi_n|^2} = \frac{\sigma(I)}{|I|}. \tag{5.27}$$

Combining (5.26) and (5.27) with (5.22), we arrive at

$$\frac{\mu(I)}{|I|} \leqslant \log\left(\frac{\sigma(I)}{|I|}\right).$$

It follows by Lebesgue's theorem on differentiation [12, 49, 53] that

$$\mu' \leqslant \log \sigma' \qquad \text{a.e. on} \quad \mathbb{T}. \tag{5.28}$$

Passing to the limit in (5.18) (assuming that $n\in\Lambda'$), we obtain

$$-\infty < \log\frac{1}{k^2} + v_s(\mathbb{T}) = \int_{\mathbb{T}} d\mu + v_s(\mathbb{T}) = \int_{\mathbb{T}} \mu'\, dm \leqslant \int_{\mathbb{T}} \log\sigma'\, dm. \tag{5.29}$$

Combining (5.29) with (5.19), we conclude that

$$\int_{\mathbb{T}} \log\sigma'\, dm = \log\frac{1}{k^2}$$

and that $\nu_s(\mathbb{T})=0$. Moreover, taking (5.28) into account, we obtain

$$\mu' = \log \sigma' \qquad \text{a.e. on} \quad \mathbb{T}. \tag{5.30}$$

It follows from (5.24) that

$$d\mu = \log \sigma' \, dm = (\omega' - \nu') \, dm.$$

Since ω was an arbitrary $(*)$-limit point of $(\mu_n^+)_{n\in\Lambda}$, this implies that $(*)-\lim_{n\in\Lambda} d\mu_n^+ = \omega' \, dm$. Since ν was an arbitrary $(*)$-limit point of $(\mu_n^-)_{n\geqslant 0}$, we conclude that

$$(*)\text{-}\lim_n d\mu_n = (\log \sigma') \, dm. \quad \blacksquare$$

Corollary 5.6. *A probability measure σ is a Szegő measure if and only if*

$$(*)\text{-}\lim_n \log \frac{1}{|\varphi_n|^2} \, dm = \log \sigma' \, dm. \tag{5.31}$$

The following corollary shows that in (5.31) we actually have convergence in the strong topology.

Corollary 5.7. *A probability measure σ is a Szegő measure if and only if*

$$\lim_n \int_{\mathbb{T}} \left| \log \frac{1}{|\varphi_n|^2} - \log \sigma' \right| dm = 0. \tag{5.32}$$

Proof. By Theorem 2, see (2.12), we have

$$\log(|\varphi_n|^2 \sigma') = \log(1 - |f_n|^2) - 2 \log |1 - \zeta b_n f_n|,$$

which implies that

$$\begin{aligned}
&\left| \log \frac{1}{|\varphi_n|^2} - \log \sigma' \right| \\
&\quad = \log^+ |\varphi_n|^2 \sigma' + \log^- |\varphi_n|^2 \sigma' \\
&\quad \leqslant \log \frac{1}{1 - |f_n|^2} + 2 \log^- |1 - \zeta b_n f_n| + 2 \log^+ |1 - \zeta b_n f_n|.
\end{aligned}$$

Since the mean value of $\log |1-\zeta b_n f_n|$ is zero, we obtain that

$$\begin{aligned}
&\int_{\mathbb{T}} \left| \log \frac{1}{|\varphi_n|^2} - \log \sigma' \right| dm \\
&\quad \leqslant \int_{\mathbb{T}} \log \frac{1}{1-|f_n|^2}\, dm + 4 \int_{\mathbb{T}} \log^+ |1-\zeta b_n f_n|\, dm \\
&\quad \leqslant \int_{\mathbb{T}} \log \frac{1}{1-|f_n|^2}\, dm + 4 \int_{\mathbb{T}} |f_n|\, dm \\
&\quad \leqslant \int_{\mathbb{T}} \log \frac{1}{1-|f_n|^2}\, dm + 4 \left(\int_{\mathbb{T}} \log \frac{1}{1-|f_n|^2}\, dm \right)^{1/2}.
\end{aligned}$$

By Szegő's theorem and (2.11), we obtain (5.32). ∎

Corollary 5.8. *Let σ be a Szegő measure. Then for every α, $0<\alpha\leqslant 1$,*

$$\lim_n \int_{\mathbb{T}} (|\varphi_n|^2 \sigma')^\alpha\, dm = 1. \tag{5.33}$$

Proof. Jensen's inequality implies

$$\exp \left\{ \int_{\mathbb{T}} \alpha \log(|\varphi_n|^2 \sigma')\, dm \right\} \leqslant \int_{\mathbb{T}} (|\varphi_n|^2 \sigma')^\alpha\, dm \leqslant 1. \tag{5.34}$$

Now (5.33) follows by Corollary 5.6. ∎

Corollary 5.9. *Let σ be a Szegő measure. Then*

$$\lim_n \int_{\mathbb{T}} |\varphi_n|^2\, d\sigma_s = 0. \tag{5..35}$$

Proof. Applying Corollary 5.8 with $\alpha=1$, we obtain that

$$\lim_n \int_{\mathbb{T}} |\varphi_n|^2\, d\sigma_s = 1 - \lim_n \int_{\mathbb{T}} |\varphi_n|^2 \sigma'\, dm = 0$$

for any measure σ satisfying (5.33) [45]. ∎

Given a Szegő measure σ we define the *Szegő function* of σ by

$$D(z) = D(\sigma, z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log \sqrt{\sigma'}\, dm(\zeta) \right). \tag{5.36}$$

Equivalently the Szegő function can be defined as the outer function in $\mathbb{D}$ [12, Chap. II, Sect. 4] satisfying $|D|^2 = \sigma'$ a.e. on $\mathbb{T}$ and $D(0) > 0$.

The following corollary is a central point of Szegő's theory. The proof is standard and makes use of the simplest form of weak and strong arguments. As usual we assume that $D^{-1} \equiv 0$ in $L^2(d\sigma_s)$.

COROLLARY 5.10. *Let σ be a Szegő measure. Then*

$$\lim_n \int_{\mathbb{T}} |\varphi_n^* - D^{-1}|^2 \, d\sigma = 0. \tag{5.37}$$

Proof. We have by (5.35) that

$$\int_{\mathbb{T}} |\varphi_n^* - D^{-1}|^2 \, \sigma' \, dm = 2 - 2 \operatorname{Re} \int_{\mathbb{T}} \varphi_n^* D \, dm + o(1) \to 0,$$

since by Corollary 5.6 and by the mean-value theorem $\lim_n \varphi_n^* D(0) = 1$. ▮

A parallel corollary can also be proved with the method of weak and strong convergence.

COROLLARY 5.11. *Let σ be a Szegő measure with $\sigma_s \equiv 0$. Then*

$$\lim_n \int_{\mathbb{T}} \left| \frac{1}{\varphi_n^*} - D \right|^2 dm = 0. \tag{5.38}$$

Proof. By (5.11) $1/\varphi_n^*$ is a point of the unit sphere of $L^2(\mathbb{T})$. Next,

$$\frac{1}{\varphi_n^*(z)} \rightrightarrows D(z) \tag{5.39}$$

uniformly on compact subsets of $\mathbb{D}$ by Corollary 5.6, which implies that $1/\varphi_n^* \to D$ in the weak topology of $L^2(\mathbb{T})$. It follows that

$$\int_{\mathbb{T}} \left| \frac{1}{\varphi_n^*} - D \right|^2 dm = 2 - 2 \operatorname{Re} \int_{\mathbb{T}} \bar{D} \cdot \frac{1}{\varphi_n^*} \, dm \to 0. \quad ▮$$

By (5.17) and (5.2) we obtain the following corollary.

COROLLARY 5.12. *A probability measure σ is a Szegő measure if and only if the Schur parameters $(\gamma_n)_{n \geqslant 0}$ of the Schur function f of σ satisfy*

$$\sum_{n=0}^{\infty} |\gamma_n|^2 < +\infty. \tag{5.40}$$

In Section 4 we proved that the Schur parameters of A_n^*/B_n are given by (4.18). The following lemma shows that A_n^*/B_n is a good approximant for the finite Blaschke product $b_{n+1}=\varphi_{n+1}/\varphi_{n+1}^*$.

LEMMA 5.13. *Let $(\varphi_n)_{n\geqslant 0}$ be orthogonal polynomials and $(A_n)_{n\geqslant 0}$, $(B_n)_{n\geqslant 0}$ the corresponding Wall polynomials. Then*

$$b_{n+1}=-\frac{A_n^*}{B_n}+\frac{\omega_n z^{n+1}}{B_n(B_n-zA_n)}. \tag{5.41}$$

Proof. By (5.5) we have

$$\begin{aligned} b_{n+1}&=\frac{\varphi_{n+1}}{\varphi_{n+1}^*}=\frac{zB_n^*-A_n^*}{B_n-zA_n}+\frac{A_n^*}{B_n}-\frac{A_n^*}{B_n}=-\frac{A_n^*}{B_n}+\frac{zB_n^*B_n-zA_n^*A_n}{B_n(B_n-zA_n)}\\ &=-\frac{A_n^*}{B_n}+\frac{\omega_n\cdot z^{n+1}}{B_n(B_n-zA_n)};\end{aligned}$$

see (4.14). ∎

In the following theorem, we extend the asymptotic formula (5.39) to the class of Rakhmanov measures.

THEOREM 5.14. *Let σ be a probability measure on $\mathbb{T}$ with Schur function f. Then σ is a Rakhmanov measure if and only if*

$$\frac{1}{\varphi_{n+1}^*(z)}=\frac{1+o(1)}{\sqrt{\omega_n}\,(1-zf)}\cdot\prod_{\kappa=0}^{n}(1-\bar{\gamma}_\kappa f_\kappa) \tag{5.42}$$

uniformly on compact subsets of $\mathbb{D}$.

Proof. By (5.5) and (4.35) we have

$$\frac{\sqrt{\omega_n}}{\varphi_{n+1}^*}=\frac{\omega_n}{B_n}\cdot\frac{1}{1-zA_n/B_n}=\frac{1+z(A_n^*/B_n)\,f_{n+1}}{1-zA_n/B_n}\cdot\prod_{\kappa=0}^{n}(1-\bar{\gamma}_\kappa f_\kappa).$$

It follows by Wall's theorem (see Corollary 4.7) that (5.42) holds if and only if $(A_n^*/B_n)\,f_{n+1}\rightrightarrows 0$ in $\mathbb{D}$, which is equivalent by Lemma 5.13 to $b_{n+1}f_{n+1}\rightrightarrows 0$. Now the result follows by Theorem 3, see (2.17). ∎

To show that (5.42) is an extension of (5.39) we need the following theorem.

THEOREM 5.15 *Let $(f_n)_{n \geqslant 0}$ be the Schur functions of $f \in \mathscr{B}$ and $(\gamma_n)_{n \geqslant 0}$ the Schur parameters of f. Then the series*

$$\sum_{n=0}^{\infty} \bar{\gamma}_n f_n(z) \tag{5.43}$$

converges uniformly on compact subsets of $\mathbb{D}$ if and only if the sequence $(\gamma_n)_{n \geqslant 0}$ satisfies (5.40); *i.e., f is the Schur function of a Szegő measure.*

Proof. If (5.43) converges in $\mathbb{D}$, then it converges at $z=0$ and we obviously obtain (5.40), which implies that f is the Schur function of a Szegő measure by Corollary 5.12.

Suppose now that $(\gamma_n)_{n \geqslant 0}$ satisfies (5.40). Let $|z| \leqslant 1-\varepsilon$, where $\varepsilon > 0$. Applying Theorem 4.13 to f_n, we obtain by (4.33.1), (4.35), and (4.36) that

$$f_n(z) = \sum_{\kappa=0}^{\infty} \gamma_{n+\kappa} z^{\kappa} \cdot h_{n,\kappa}(z), \tag{5.44}$$

where $|h_{n,\kappa}(z)| \leqslant \varepsilon^{-1}$ in $|z| \leqslant 1-\varepsilon$. It follows that

$$\begin{aligned} \sum_{n=\mathcal{N}}^{M} |\bar{\gamma}_n f_n(z)| &\leqslant \sum_{n=\mathcal{N}}^{M} \sum_{\kappa=0}^{\infty} |\gamma_n \gamma_{n+\kappa}| \, \varepsilon^{-1} (1-\varepsilon)^{\kappa} \\ &\leqslant \left(\sum_{n \geqslant \mathcal{N}} |\gamma_n|^2 \right) \cdot \varepsilon^{-2} \end{aligned} \tag{5.45}$$

for $|z| \leqslant 1-\varepsilon$, which implies (5.43). ▮

The infinite product $\prod (1-\bar{\gamma}_n f_n)$ converges absolutely if and only if the series $\sum |\gamma_n f_n|$ converges. Suppose that σ is a Szegő measure. Then by (5.42) and (5.39) we obtain that

$$D(\sigma, z) = \frac{1}{\sqrt{\omega}(1-zf)} \cdot \prod_{n=0}^{\infty} (1-\bar{\gamma}_n f_n), \qquad z \in \mathbb{D}, \tag{5.46}$$

or, equivalently, by (4.26) that

$$\frac{\sqrt{\omega}}{D(\sigma, z)} = \prod_{n=0}^{\infty} (1 - z\bar{\gamma}_{n-1} f_n), \qquad z \in \mathbb{D}, \tag{5.47}$$

where $\omega = \lim_n \omega_n$, $\gamma_{-1} = -1$.

6. ERDŐS MEASURES

We begin our proof of Theorem 1 with a proof of Theorem 2.

Proof of Theorem 2. By (4.19) and (4.14) we have for $\zeta \in \mathbb{T}$

$$1-|f(\zeta)|^2 = 1 - \left|\frac{A_{n-1}+\zeta B^*_{n-1} f_n}{B_{n-1}+\zeta A^*_{n-1} f_n}\right|^2 = \frac{(1-|f_n(\zeta)|^2)\,\omega_{n-1}}{|B_{n-1}+\zeta A^*_{n-1} f_n|^2}. \tag{6.1}$$

Notice that by (5.2) $k_n^{-2} = \omega_{n-1}$. Therefore, it follows from (4.19) and (5.5) that

$$|1-\zeta f|^2 = \left|1 - \frac{\zeta A_{n-1}+\zeta^2 B^*_{n-1} f_n}{B_{n-1}+\zeta A^*_{n-1} f_n}\right|^2$$

$$= \omega_{n-1} \frac{|\varphi^*_n - \zeta \varphi_n f_n|^2}{|B_{n-1}+\zeta A^*_{n-1} f_n|^2}. \tag{6.2}$$

Now by (2.2) we obtain from (6.1) and (6.2) that

$$\sigma' = \frac{1-|f|^2}{|1-\zeta f|^2} = \frac{1-|f_n|^2}{|\varphi^*_n - \zeta \varphi_n f_n|^2} \tag{6.3}$$

a.e. on $\mathbb{T}$. Multiplying (6.3) by $|\varphi_n|^2 = |\varphi^*_n|^2$, we obtain (2.12). ∎

Let us multiply (2.12) by $|1-\zeta b_n f_n|^2 = 1+|f_n|^2 - 2\,\mathrm{Re}(\zeta b_n f_n)$. After simple algebra we obtain

$$|f_n|^2 = \frac{1-|\varphi_n|^2\sigma'}{1+|\varphi_n|^2\sigma'} + \mathrm{Re}(\zeta b_n f_n) + \frac{|\varphi_n|^2\sigma' - 1}{1+|\varphi_n|^2\sigma'} \cdot \mathrm{Re}(\zeta b_n f_n). \tag{6.4}$$

The mean-value theorem yields

$$\int_{\mathbb{T}} \mathrm{Re}(\zeta b_n f_n)\, dm = \mathrm{Re} \int_{\mathbb{T}} \zeta b_n f_n\, dm = 0. \tag{6.5}$$

Therefore, we obtain from (6.4) that

$$\int_{\mathbb{T}} |f_n|^2\, dm \leqslant 2 \int_{\mathbb{T}} \left|1 - \frac{2\,|\varphi_n|^2\sigma'}{1+|\varphi_n|^2\sigma'}\right| dm. \tag{6.6}$$

Now the proof of Theorem 1 can be completed by Corollary 2.2 of [38], since obviously

$$\int_{\mathbb{T}} \left| 1 - \frac{2\,|\varphi_n|^2\,\sigma'}{1+|\varphi_n|^2\,\sigma'} \right| dm \leqslant \int_{\mathbb{T}} |1 - |\varphi_n|^2\,\sigma'|\, dm. \tag{6.7}$$

However, we provide another proof, different parts of which will be generalized later.

THEOREM 6.1. *Let σ be an Erdős measure and let $(\varphi_n)_{n \geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then*

$$\lim_n \int_{\mathbb{T}} \left| 1 - \frac{2\,|\varphi_n|^2\,\sigma'}{1+|\varphi_n|^2\,\sigma'} \right|^2 dm = 0. \tag{6.8}$$

Proof. We consider on $\mathbb{T}$ and auxiliary sequence of functions

$$g_n = \frac{2\,|\varphi_n|^2\,\sigma'}{1+|\varphi_n|^2\,\sigma'}, \qquad n = 0, 1, ... \tag{6.9}$$

It is clear that $0 \leqslant g_n < 2$ a.e. on $\mathbb{T}$.

LEMMA 6.2. *Let Φ be a function on $[0, +\infty)$ defined by*

$$\Phi(x) = \begin{cases} x, & 0 \leqslant x < 1, \\ \dfrac{4x^2}{(1+x)^2}, & 1 \leqslant x < +\infty. \end{cases} \tag{6.10}$$

Then Φ is an increasing concave function on $[0, +\infty)$ satisfying

$$\frac{4x^2}{(1+x)^2} \leqslant \Phi(x), \qquad 0 \leqslant x < +\infty. \tag{6.11}$$

Proof. Simple calculus shows that for $1 \leqslant x < +\infty$ we have

$$\Phi'(x) = 8x(1+x)^{-3}; \qquad \Phi''(x) = 8(1-2x)(1+x)^{-4}.$$

since $\Phi' \equiv 1$ on $(0, 1)$, $\lim_{x \to 1+0} \Phi'(x) = 1$, and $\Phi''(x) < 0$ on $(1, +\infty)$, it follows that Φ is an increasing and concave function on $[0, +\infty)$. Finally, (6.11) turns into equality for $x \geqslant 1$ and is elementary for $x < 1$. ∎

Taking into account (6.11) and applying Jensen's inequality, we obtain that

$$\int_{\mathbb{T}} g_n^2\,dm = \int_{\mathbb{T}} \frac{4(|\varphi_n|^2\,\sigma')^2}{(1+|\varphi_n|^2\,\sigma')^2}\,dm \leqslant \int_{\mathbb{T}} \Phi(|\varphi_n|^2\,\sigma')\,dm$$
$$\leqslant \Phi\left(\int_{\mathbb{T}} |\varphi_n|^2\,\sigma'\,dm\right) \leqslant \Phi(1) = 1. \tag{6.12}$$

It follows that

$$\int_{\mathbb{T}} g_n\,dm \leqslant \left(\int_{\mathbb{T}} g_n^2\,dm\right)^{1/2} \leqslant 1. \tag{6.13}$$

On the other hand, for any open arc I on $\mathbb{T}$, we have by Cauchy's inequality

$$\begin{aligned}\frac{1}{|I|}\int_I \sqrt{\sigma'}\,dm &= \frac{1}{|I|}\int_I \frac{\sqrt{2}\,|\varphi_n|\,\sqrt{\sigma'}}{(1+|\varphi_n|^2\,\sigma')^{1/2}} \cdot \frac{(1+|\varphi_n|^2\,\sigma')^{1/2}}{\sqrt{2}\,|\varphi_n|}\,dm \\ &\leqslant \left(\frac{1}{|I|}\int_I g_n\,dm\right)^{1/2} \cdot \left(\frac{1}{2\,|I|}\int_I \left(\frac{1}{|\varphi_h|^2} + \sigma'\right) dm\right)^{1/2}.\end{aligned} \tag{6.14}$$

Let g be *an arbitrary* weak-($*$) limit point of the bounded sequence $(g_n)_{n\geqslant 0}$ in $L^\infty(\mathbb{T})$, i.e., a limit point in the topology induced by the standard duality (L^1, L^∞). Suppose that σ has no mass at the end-points of I. Passing in (6.14) to the limit and applying Helly's theorem and (5.15), we obtain that the inequality

$$\frac{1}{|I|}\int_I \sqrt{\sigma'}\,dm \leqslant \left(\frac{1}{|I|}\int_I g\,dm\right)^{1/2} \left(\frac{1}{2}\cdot\frac{\sigma(I)}{|I|} + \frac{1}{2\,|I|}\int_I \sigma'\,dm\right)^{1/2} \tag{6.15}$$

holds for any open arc I on $\mathbb{T}$ except possibly for a family of arcs with endpoints carrying point masses of σ. Now we apply Lebesgue's theorem on differentiation to (6.15) and obtain that

$$\sqrt{\sigma'} \leqslant \sqrt{g}\cdot(\tfrac{1}{2}\sigma' + \tfrac{1}{2}\sigma')^{1/2}$$

a.e. on $\mathbb{T}$. Since $\sigma' > 0$ a.e. on $\mathbb{T}$, this implies that $g \geqslant 1$ a.e. on $\mathbb{T}$. Combining this last inequality with (6.13) and observing that g is an arbitrary limit point of $(g_n)_{n\geqslant 0}$ in the weak-($*$) topology, we obtain that the following limits exist and are equal:

$$\lim_n \int_{\mathbb{T}} g_n\,dm = \lim_n \int_{\mathbb{T}} g_n^2\,dm = 1. \tag{6.16}$$

It follows that

$$\lim_n \int_{\mathbb{T}} (1-g_n)^2\, dm = 1 - 2 \lim_n \int_{\mathbb{T}} g_n\, dm + \lim_n \int_{\mathbb{T}} g_n^2\, dm = 0,$$

which obviously implies (6.8).

Remark. The idea of applying inequalities of type (6.14) to the proof of weak-($*$) convergence originates in paper by Rakhmanov [45, Lemma 2]. However, here it is used in a different context. Besides, instead of the construction of [45] we use Lebesgue's theorem on differentiation.

Proof of Theorem 1. If $m\{\zeta \in \mathbb{T} : |f(\zeta)| = 1\} = \delta > 0$, then by (6.1) $m\{\zeta \in \mathbb{T} : |f_n(\zeta)| = 1\} = \delta$ for every n and therefore (2.1) cannot hold.
If $|f| < 1$ a.e. on $\mathbb{T}$, then (2.1) follows from (6.6) by Theorem 6.1. ▮

The following theorem extends Theorem 6.1 to the class of Rakhmanov measures. Given a probability measure σ on $\mathbb{T}$ we denote by $E = E(\sigma)$ the Lebesgue support $\{\zeta \in \mathbb{T} : \sigma'(\zeta) > 0\}$ of the absolutely continuous part of σ.

THEOREM 6.3. *Let σ be a Rakhmanov measure and let $(\varphi_n)_{n \geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then*

$$\lim_n \int_{E(\sigma)} \left| 1 - \frac{2\, |\varphi_n|^2\, \sigma'}{1 + |\varphi_n|^2\, \sigma'} \right|^2 dm = 0. \tag{6.17}$$

Proof. Let $(g_n)_{n \geqslant 0}$ be defined by (6.9) and let Φ be defined by (6.10). For any open arc I on $\mathbb{T}$ we obtain by Jensen's inequality that

$$\frac{1}{|I|} \int_I g_n^2\, dm \leqslant \frac{1}{|I|} \int_I \Phi(|\varphi_n|^2\, \sigma')\, dm \leqslant \Phi\left(\frac{1}{|I|} \int_I |\varphi_n|^2\, d\sigma \right). \tag{6.18}$$

Since σ is a Rakhmanov measure, we obtain by Helly's theorem that

$$\overline{\lim_n} \frac{1}{|I|} \int_I g_n^2\, dm \leqslant \Phi(1) = 1 \tag{6.19}$$

and consequently

$$\overline{\lim_n} \frac{1}{|I|} \int_I g_n\, dm \leqslant 1. \tag{6.20}$$

Let g be any limit point of $(g_n)_{n\geqslant 0}$ in the weak-(∗) topology of $L^\infty(\mathbb{T})$ and let G be any limit point of $(g_n^2)_{n\geqslant 0}$ in this topology. By (6.14) we have

$$\frac{1}{|I|}\int_I \sqrt{\sigma'}\,dm \leqslant \left(\frac{1}{|I|}\int_I g_n\,dm\right)^{1/2}\left(\frac{1}{2\,|I|}\int_I\left(\frac{1}{|\varphi_n|^2}+\sigma'\right)dm\right)^{1/2},$$

which obviously implies that

$$\frac{1}{|I|}\int_I \sqrt{\sigma'}\,dm \leqslant \left(\frac{1}{|I|}\int_I g_n^2\,dm\right)^{1/4}\left(\frac{1}{2\,|I|}\int_I\left(\frac{1}{|\varphi_n|^2}+\sigma'\right)dm\right)^{1/2}.$$

Passing to the limit in these inequalities, we obtain by Helly's theorem and by Lemma 5.4 that

$$\begin{aligned}\frac{1}{|I|}\int_I \sqrt{\sigma'}\,dm &\leqslant \left(\frac{1}{|I|}\int_I g\,dm\right)^{1/2}\left(\frac{\sigma(I)}{2\,|I|}+\frac{1}{2\,|I|}\int_I \sigma'\,dm\right)^{1/2},\\ \frac{1}{|I|}\int_I \sqrt{\sigma'}\,dm &\leqslant \left(\frac{1}{|I|}\int_I G\,dm\right)^{1/4}\left(\frac{\sigma(I)}{2\,|I|}+\frac{1}{2\,|I|}\int_I \sigma'\,dm\right)^{1/2},\end{aligned} \tag{6.21}$$

for every open arc I whose endpoints do not carry mass of σ. By Lebesgue's theorem on differentiation we obtain from (6.21) that

$$\sqrt{\sigma'} \leqslant \sqrt{g}\cdot\sqrt{\sigma'}, \qquad \sqrt{\sigma'} \leqslant \sqrt[4]{G}\cdot\sqrt{\sigma'},$$

a.e. on $\mathbb{T}$. It follows that

$$1 \leqslant \min(g, G), \qquad \text{a.e. on } E(\sigma). \tag{6.22}$$

On the other hand, passing to the limit in (6.19) and (6.20), we obtain

$$\frac{1}{|I|}\int_I g\,dm \leqslant 1; \qquad \frac{1}{|I|}\int_I G\,dm \leqslant 1 \tag{6.23}$$

for any open arc I. Applying Lebesgue's theorem on differentiation, we conclude that

$$\max(g, G) \leqslant 1 \qquad \text{a.e. on } \mathbb{T}. \tag{6.24}$$

Obviously $g_n = g_n^2 = 0$ on $\mathbb{T}\backslash E(\sigma)$, which implies that $g \equiv G \equiv 0$ on $\mathbb{T}\backslash E(\sigma)$. It follows from (6.22) and (6.24) that

$$g = G = \mathbb{1}_E,$$

where $\mathbb{1}_E$ denotes the indicator of the set $E = E(\sigma)$.

Since g and G were chosen to be arbitrary weak-($*$) limit points of $(g_n)_{n\geqslant 0}$ and $(g_n^2)_{n\geqslant 0}$, respectively, we may conclude that both these sequences converge to $\mathbb{1}_E$ in the weak-($*$) topology of $L^\infty(\mathbb{T})$. This implies that

$$\int_E (1-g_n)^2\,dm = |E| + \int_{\mathbb{T}} g_n^2\,dm - 2\int_{\mathbb{T}} g_n\,dm \to 0, \qquad n\to+\infty. \blacksquare$$

THEOREM 6.4. *Let σ be a probability measure on $\mathbb{T}$ and let $(\varphi_n)_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Let f be the Schur function of σ. Then*

$$\int_{\mathbb{T}} \big|\,|\varphi_n|^2\,\sigma' - 1\big|\,dm \leqslant 12\cdot\int_{\mathbb{T}} |f_n|\,dm, \qquad n=0,1,\ldots. \tag{6.25}$$

Proof. It follows from (2.12) that

$$(|\varphi_n|^2\,\sigma'-1)\cdot|1-\zeta b_n f_n|^2 = 2(\operatorname{Re}(\zeta b_n f_n) - |f_n|^2) \tag{6.26}$$

a.e. on $\mathbb{T}$. Let ζ be a point of $\mathbb{T}$ such that

$$(|\varphi_n|^2\,\sigma'-1) = -(|\varphi_n|^2\,\sigma'-1)_- < 0.$$

By (6.26) we obtain that $\operatorname{Re}(\zeta b_n f_n) < |f_n|^2$ and therefore

$$|1-\zeta b_n f_n| \geqslant 1 - \operatorname{Re}(\zeta b_n f_n) > 1 - |f_n|^2.$$

Since $|f_n|\leqslant 1$ and $|\operatorname{Re}\zeta b_n f_n| \leqslant |f_n|$ a.e. on $\mathbb{T}$, we obtain from (6.26) that

$$(|\varphi_n|^2\,\sigma'-1)_-\,(1-|f_n|^2)^2 \leqslant 2\,|f_n| + 2\,|f_n|^2. \tag{6.27}$$

Since $|\varphi_n|^2\,\sigma' \geqslant 0$, it is clear that $(|\varphi_n|^2\,\sigma'-1)_- \leqslant 1$. It follows that

$$(|\varphi_n|^2\,\sigma'-1)_- \leqslant 2\,|f_n| + 2\,|f_n|^2 + 2\,|f_n|^2 \leqslant 6\cdot|f_n|, \tag{6.28}$$

which implies that

$$\int_{\mathbb{T}} (|\varphi_n|^2\,\sigma'-1)_-\,dm \leqslant 6\cdot\int_{\mathbb{T}} |f_n|\,dm. \tag{6.29}$$

Finally,

$$\int_{\mathbb{T}} (|\varphi_n|^2\,\sigma'-1)_+\,dm = \int_{\mathbb{T}} (|\varphi_n|^2\,\sigma'-1)\,dm + \int_{\mathbb{T}} (|\varphi_n|^2\,\sigma'-1)_-\,dm$$

$$\leqslant 6\int_{\mathbb{T}} |f_n|\,dm,$$

since $\int |\varphi_n|^2\,\sigma'\,dm \leqslant \int |\varphi_n|^2\,d\sigma = 1$. $\blacksquare$

It is known [15] (see Corollary 5.10) that $\lim_n \| |\varphi_n|^2 \sigma' - 1 \|_{L^1(d\sigma)} = 0$ for any Szegő measure. This was extended to Erdős measures in [38] (see [47] for another proof). However, for Szegő measures the polynomials φ_n^* converge in $L^2(d\sigma)$ and therefore converge in measure with respect to the Lebesgue measure m. On the other hand, if some subsequence of $(\varphi_n^*)_{n \geqslant 0}$ converges in measure on some measurable subset E, $E \subset \mathbb{T}$, $mE > 0$, to an almost everywhere finite measurable function, then σ is a Szegő measure [15, Theorem 5.9]. This result follows from the observation that all functions $(\varphi_n^*)^{-1}$ belong to the unit ball of the Hardy class H^2 and from the Khinchin–Ostrovskii theorem [44].

From this point of view the results obtained in [38] say that although it is meaningless to talk about the convergence of φ_n^* on $\mathbb{T}$ if σ is not a Szegő measure, there are no obstacles to the convergence of $|\varphi_n^*| = |\varphi_n|$ on $\mathbb{T}$ if σ is an Erdős measure.

In the following theorem we show how one can deduce the basic results of [38] from the results of the present section.

THEOREM 6.5. *Let σ be a probability measure on $\mathbb{T}$ with Schur function f and $(\varphi_n)_{n \geqslant 0}$ the orthogonal polynomials in $L^2(d\sigma)$. Then the following statements are equivalent*:

(1) *σ is an Erdős measure*;

(2) *the sequence $(f_n)_{n \geqslant 0}$ of the Schur functions of f converges to* 0 *in measure on $\mathbb{T}$ (with respect to m)*;

(3)

$$\lim_n \int_{\mathbb{T}} \left| |\varphi_n|^2 \sigma' - 1 \right| dm = 0;$$

(4) *there exists* α, $0 < \alpha < 1$, *such that*

$$\lim_n \int_{\mathbb{T}} (|\varphi_n|^2 \sigma')^\alpha \, dm = 1; \tag{6.30}$$

(5) (6.30) *holds for every* α, $0 < \alpha \leqslant 1$.

Proof. (1) $\Rightarrow$ (2) by Theorem 1.

(2) $\Rightarrow$ (3) by Theorem 6.4.

(3) $\Rightarrow$ (4) We prove (6.30) for $\alpha = \frac{1}{2}$. Using an elementary inequality

$$|\sqrt{a} - \sqrt{b}| \leqslant \sqrt{|a-b|}, \qquad a, b > 0,$$

we obtain by Jensen's inequality

$$\int_{\mathbb{T}} \left| |\varphi_n| \sqrt{\sigma'} - 1 \right| dm \leqslant \int_{\mathbb{T}} \sqrt{\left| |\varphi_n|^2 \sigma' - 1 \right|}\, dm$$
$$\leqslant \sqrt{\int_{\mathbb{T}} \left| |\varphi_n|^2 \sigma' - 1 \right| dm} \to 0.$$

(4) ⇒ (5) We put

$$\beta_n(\alpha) = \int_{\mathbb{T}} (|\varphi_n|^2 \sigma')^\alpha \, dm, \qquad 0 < \alpha \leqslant 1.$$

The function β_n is logarithmically convex [56, Theorem 10.12], while $\alpha \to \beta_n(d)^{1/\alpha}$ is increasing on (0, 1]. Since obviously $\beta_n(1) \leqslant 1$, we obtain that $\lim_n \beta_n(\alpha) = 1$ if $\lim_n \beta_n(\alpha_0) = 1$ and $\alpha_0 \leqslant \alpha \leqslant 1$. Now, let $0 < \alpha < \alpha_0 < 1$, $\lim_n \beta_n(\alpha_0) = 1$. Then $\alpha_0 = \alpha t_0 + t_1$, where $t_0 + t_1 = 1$, $t_i > 0$. The logarithmic convexity of β_n implies

$$\beta_n(\alpha_0) \leqslant \beta_n(\alpha)^{t_0} \cdot \beta_n(1)^{t_1},$$

which yields $\lim_n \beta_n(\alpha) = 1$.

(5) ⇒ (1) Applying the elementary identity $|a - b| = |\sqrt{a} - \sqrt{b}| \cdot |\sqrt{a} + \sqrt{b}|$ and Cauchy's inequality, we obtain

$$\int_{\mathbb{T}} \left| |\varphi_n|^2 \sigma' - 1 \right| dm \leqslant 2 \left(\int_{\mathbb{T}} (|\varphi_n| \sqrt{\sigma'} - 1)^2 \, dm \right)^{1/2}$$
$$= 2(1 + \beta_n(1) - 2\beta_n(\tfrac{1}{2}))^{1/2} \to 0,$$

which obviously implies that $m\{\zeta \in \mathbb{T} : \sigma'(\zeta) = 0\} = 0$. ∎

7. RAKHMANOV MEASURES

Proof of Theorem 3. Let us suppose fist that σ is absolutely continuous; i.e., $d\sigma = \sigma' \, dm$. By Fatou's theorem on nontangential limits [12, Chap. I, Theorem 5.3] we have

$$\operatorname{Re} \frac{1 + z b_n f_n}{1 - z b_n f_n} = \frac{1 - |f_n|^2}{|1 - z b_n f_n|^2}$$

a.e. on $\mathbb{T}$. By (2.12) we obtain

$$|\varphi_n|^2\,\sigma' = \mathrm{Re}\,\frac{1+zb_n f_n}{1-zb_n f_n} \qquad \text{a.e. on } \mathbb{T}. \tag{7.1}$$

The function $z \mapsto (1+zb_n f_n)/(1-zb_n f_n)$ is holomorphic in $\mathbb{D}$, it equals 1 at $z=0$, and its real part is non-negative in $\mathbb{D}$. By Herglotz' theorem this function is represented as the Schwarz integral of a probability measure μ. It follows from (7.1) that $\mu' = |\varphi_n|^2\,\sigma'$ a.e. on $\mathbb{T}$. Since we assumed that $d\sigma = \sigma'\,dm$, we obviously obtain

$$\int_{\mathbb{T}} |\varphi_n|^2\,\sigma'\,dm = \int_{\mathbb{T}} |\varphi_n|^2\,d\sigma = 1.$$

It follows that $d\mu = \mu'\,dm = |\varphi_n|^2\,\sigma'\,dm$, which completes the proof for absolutely continuous measures σ.

Now, let σ be an arbitrary probability measure on $\mathbb{T}$. By (5.13) the measures $|\varphi_n|^{-2}\,dm$ and $d\sigma$ have identical Fourier coefficients for the indices κ, $|\kappa| \leqslant n$. This implies that the polynomials $\varphi_0, \varphi_1, ..., \varphi_n$ are orthogonal both in $L^2(d\sigma)$ and in $L^2(|\varphi_n|^{-2}\,dm)$.

Applying Theorem 3 to the absolutely continuous measure $|\varphi_{n+\kappa}|^{-1}\,dm$, $\kappa = 0, 1, ...,$ and using (5.11), we obtain that

$$\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\,|\varphi_n(\zeta)|^2\,\frac{dm}{|\varphi_{n+\kappa}|^2} = \frac{1+zb_n g_n^\kappa}{1-zb_n g_n^\kappa}, \qquad |z|<1, \tag{7.2}$$

where g_n^κ are the Schur functions of order n of $A_{n+\kappa-1}/B_{n+\kappa-1}$. By Wall's theorem (see Corollary 4.7)

$$\lim_{\kappa} \frac{A_{n+\kappa-1}}{B_{n+\kappa-1}} = f$$

uniformly on compact subsets of $\mathbb{D}$. By Lemma 4.11 we obtain that

$$g_n^\kappa(z) \rightrightarrows f_n(z), \qquad \kappa \to \infty, \quad |z|<1. \tag{7.3}$$

Taking (7.3) and (5.15) into account, we obtain (2.14) by passing to the limit in (7.2) as $\kappa \to +\infty$. ∎

The following immediate consequence of Theorem 3 will be used in the proof of Theorem 5.

COROLLLARY 7.1. *A probability measure σ on $\mathbb{T}$ is a Rakhmanov measure if and only if*

$$f_n b_n \rightrightarrows 0, \qquad n \to +\infty \tag{7.4}$$

uniformly on compact subsets of $\mathbb{D}$.

COROLLARY 7.2. *Let σ be a probability measure with Geronimus parameters $(a_n)_{n\geqslant 0}$ and $(\varphi_n)_{n\geqslant 0}$ the orthogonal polynomials in $L^2(d\sigma)$. Then*

$$\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\left|\frac{\varphi_n}{\varphi_{n+1}}\right|^2 dm = \frac{1+za_nb_n(z)}{1-za_nb_n(z)}, \qquad |z|<1, \tag{7.5}$$

$$\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\frac{1}{2}\left(1+\left|\frac{\varphi_n}{\varphi_{n+1}}\right|^2\right) dm = \frac{1}{1-a_nzb_n(z)} = \frac{\Phi_n^*(z)}{\Phi_{n+1}^*(z)}, \qquad |z|<1. \tag{7.6}$$

Proof. Since the Schur function of order n for A_n/B_n is the constant a_n (see (4.17) and Geronimus' theorem), (7.5) follows from (2.14). Indeed, by (5.11) the rational function A_n/B_n is the Schur function of the probability measure $|\varphi_{n+1}|^{-2}\, dm$. Finally, (7.6) is immediate from (7.5). ∎

Remark. Compare (7.5) and (7.6) with Lemma 4 by Rakhmanov [45].

Proof of Theorem 4. Let us suppose that $(a_n)_{n\geqslant 0}$ satisfies (2.18) for every κ, $\kappa=1, 2, \ldots$. We have

$$\int_{\mathbb{T}} |\zeta\varphi_n - \varphi_{n+1}|^2\, d\sigma = 2(1-\sqrt{1-|a_n|^2}) \leqslant 2\,|a_n|^2. \tag{7.7}$$

Since obviously $\zeta^\kappa\varphi_n \perp \zeta^\kappa\varphi_{n-\kappa}$, $\varphi_{n+\kappa}\perp\zeta^\kappa\varphi_{n-\kappa}$, $\varphi_{n+\kappa}\perp\varphi_n$, $\kappa=1, 2, \ldots,$ we obtain

$$\int_{\mathbb{T}} \zeta^\kappa\,|\varphi_n|^2\, d\sigma = -\int_{\mathbb{T}} (\zeta^\kappa\varphi_n - \varphi_{n+\kappa})\overline{(\zeta^\kappa\varphi_{n-\kappa}-\varphi_n)}\, d\sigma \tag{7.8}$$

for $\kappa = 1, 2, \ldots$. The following identities are obvious:

$$\begin{aligned}(\zeta^\kappa\varphi_n - \varphi_{n+\kappa}) &= (\zeta^\kappa\varphi_n - \zeta^{\kappa-1}\varphi_{n+1}) + (\zeta^{\kappa-1}\varphi_{n+1}-\zeta^{\kappa-2}\varphi_{n+2}) + \cdots \\ &\quad + (\zeta\varphi_{n+\kappa-1}-\varphi_{n+\kappa}),\\ (\zeta^\kappa\varphi_{n-\kappa}-\varphi_n) &= (\zeta^\kappa\varphi_{n-\kappa}-\zeta^{\kappa-1}\varphi_{n-\kappa+1}) \\ &\quad + (\zeta^{\kappa-1}\varphi_{n-\kappa+1}-\zeta^{\kappa-2}\varphi_{n-\kappa+2}) + \cdots + (\zeta\varphi_{n-1}-\varphi_n).\end{aligned} \tag{7.9}$$

Taking into account (7.7) and (7.9), we obtain from (7.8) by Cauchy's inequality (for $\kappa = 1, 2, ...$) that

$$\left| \int_{\mathcal{T}} \zeta^\kappa \, |\varphi_n|^2 \, d\sigma \right|$$

$$\leqslant \|\zeta^\kappa \varphi_n - \varphi_{n+\kappa}\| \cdot \|\zeta^\kappa \varphi_{n-\kappa} - \varphi_n\|$$

$$\leqslant 2(|a_n| + |a_{n+1}| + \cdots + |a_{n+\kappa-1}|)(|a_{n-\kappa}| + \cdots + |a_{n-1}|). \qquad (7.10)$$

It follows from (2.18) that the right-hand side of (7.10) tends to zero as $n \to +\infty$. Hence σ is a Rakhmanov measure.

Let us suppose now that σ is a Rakhmanov measure. Then by Corollary 7.1 $f_n b_n \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$. Taking the quotient of the recurrence formulae in (1.11), we obtain that

$$b_{n+1}(z) = \frac{z b_n(z) - \bar{a}_n}{1 - z a_n b_n(z)}. \qquad (7.11)$$

By (1.3) and (7.11) we have

$$z b_n f_n = \frac{b_{n+1} + \bar{a}_n}{1 + a_n b_{n+1}} \cdot \frac{z f_{n+1} + a_n}{1 + \bar{a}_n z f_{n+1}}.$$

It follows that

$$z b_n f_n (1 + a_n b_{n+1})(1 + \bar{a}_n z f_{n+1}) = z b_{n+1} f_{n+1} + a_n b_{n+1} + |a_n|^2 + \bar{a}_n z f_{n+1},$$

which obviously implies that

$$a_n b_{n+1}(z) + |a_n|^2 + \bar{a}_n z f_{n+1} \rightrightarrows 0. \qquad (7.12)$$

Multiplying (7.12) by f_{n+1}, we obtain that

$$\gamma_n f_{n+1}(z)(\gamma_n + z f_{n+1}(z)) \rightrightarrows 0. \qquad (7.13)$$

Notice that $a_n = \gamma_n$ by Geronimus' theorem.

Lemma 7.3. *Let f be a function in $\mathcal{B}$ satisfying* (7.13). *Then the sequence* $(\gamma_n)_{n \geqslant 0}$ *of the Schur parameters of f satisfies* (2.18).

Proof. We prove that

$$\gamma_n f_{n+\kappa}(z)(\gamma_n + z f_{n+1}(z)) \rightrightarrows 0, \qquad n \to +\infty, \qquad (7.14)$$

uniformly on compact subsets of $\mathbb{D}$ for $\kappa = 1, 2, \ldots$. For $\kappa = 1$ (7.14) coincides with (7.13). Let us suppose now that (7.14) holds for some κ and prove that it holds for $\kappa + 1$. We have

$$f_{n+\kappa}(1 + \bar{\gamma}_{n+\kappa} z f_{n+\kappa+1}) = z f_{n+\kappa+1} + \gamma_{n+\kappa}. \tag{7.15}$$

It follows from (7.14) (put $z = 0$) that

$$|\gamma_n \gamma_{n+\kappa}| \leqslant \sqrt{|\gamma_n^2 \gamma_{n+\kappa}|} \to 0, \qquad n \to +\infty.$$

Multiplying (7.15) by $\gamma_n(\gamma_n + z f_{n+1})$, we obtain from (7.14) that

$$\gamma_n f_{n+\kappa+1}(z)(\gamma_n + z f_{n+1}(z)) \rightrightarrows 0, \qquad n \to +\infty. \quad ▮$$

The Máté–Nevai condition (2.18) with $\kappa = 1$ appeared for the first time in [34], where it was shown that (2.18) for $\kappa = 1$ is a necessary condition for (2.19). An example of a measure satisfying (2.19) but which is not in Nevai's class was also given in [34].

In the following theorem we present different equivalent descriptions of Rakhmanov measures.

THEOREM 7.4. *Let σ be a probability measure on $\mathbb{T}$ with Geronimus parameters $(a_n)_{n \geqslant 0}$, f the Schur function of σ, $(\varphi_n)_{n \geqslant 0}$ the orthogonal polynomials in $L^2(d\sigma)$, $(\Phi_n)_{n \geqslant 0}$ the monic orthogonal polynomials, i.e. $\Phi_n = k_n^{-1} \cdot \varphi_n$, and $b_n = \varphi_n / \varphi_n^*$. Then the following conditions are equivalent*:

(1) *σ is a Rakhmanov measure*;

(2) *the Geronimus parameters $(a_n)_{n \geqslant 0}$ (equivalently the Schur parameters $(\gamma_n)_{n \geqslant 0}$ of f) satisfy the Máté–Nevai condition* (2.18) *for every* $\kappa = 1, 2, \ldots$;

(3) $\gamma_n f_{n+1}(z) \rightrightarrows 0$, $n \to +\infty$, *uniformly on compact subsets of* $\mathbb{D}$;

(4) $a_n b_n(z) \rightrightarrows 0$, $n \to +\infty$, *uniformly on compact subsets of* $\mathbb{D}$;

(5) $b_n(z)\, f_n(z) \rightrightarrows 0$, $n \to +\infty$, *uniformly on compact subsets of* $\mathbb{D}$;

(6) $(*) - \lim_n |\frac{\varphi_n}{\varphi_{n+1}}|^2\, dm = dm$;

(7) $(*) - \lim_n |\frac{\varphi_n}{\varphi_{n+l}}|^2\, dm = dm$ *for every* $l = 0, 1, 2, \ldots$;

(8)

$$\frac{\Phi_{n+1}^*(z)}{\Phi_n^*(z)} \rightrightarrows 1, \qquad n \to +\infty,$$

uniformly on compact subsets of $\mathbb{D}$.

Proof. (1) ⇔ (2) by Theorem 4.

(2) ⇒ (3) By (4.33.2) and (4.36) we have

$$f_{n+1} = \gamma_{n+1} + \sum_{\kappa=1}^{\infty} \gamma_{n+1+\kappa} z^{\kappa} \cdot h_{\kappa,n}(z), \qquad |h_{\kappa,n}(z)| \leqslant (1-|z|)^{-1}. \tag{7.16}$$

Multiplying (7.16) by γ_n, we obtain (3).

(3) ⇒ (2) If $\gamma_n f_{n+1} \rightrightarrows 0$, then we obviously have (7.13), which implies (2) by Lemma 7.3.

(2) ⇒ (4) is similar to (2) ⇒ (3), since the Schur parameters of b_n (see Lemma 5.13 and (4.18)) are given by a finite sequence $-\bar{a}_{n-1}, ..., -\bar{a}_0, 1$.

(4) ⇒ (2) If $a_n b_n(z) \rightrightarrows 0$, then (put $z=0$) $a_n a_{n-1} \to 0$. By (7.11) we have

$$b_n(z)(1 - z a_{n-1} b_{n-1}(z)) = z b_{n-1}(z) - \bar{a}_{n-1}. \tag{7.17}$$

Multiplying (7.17) by a_n, we obtain that $a_n b_{n-1}(z) \rightrightarrows 0$ and therefore $a_n a_{n-2} \to 0$. Now the proof is completed by induction.

(1) ⇔ (5) by Corollary 7.1.

(2) ⇔ (6) by Corollary 7.2 (see (7.5)) and by the already proved equivalence (4) ⇔ (2).

(2) ⇔ (8) by Corollary 7.2 (see (7.6)) and by the already proved equivalence (4) ⇔ (2).

(7) ⇒ (6) is obvious.

(2) ⇒ (7) By (5.11) and by (4.17) the Geronimus parameters of $|\varphi_{n+l}|^{-2}\, dm$ are given by

$$a_0, ..., a_{n+l-1}, 0, 0, ...$$

Now, we put $d\sigma = |\varphi_{n+l}|^{-2}\, dm$ in (7.10) and obtain

$$\left| \int_{\mathbb{T}} \zeta^{\kappa} \left| \frac{\varphi_n}{\varphi_{n+l}} \right|^2 dm \right| \leqslant 2(|a_n| + |a_{n+1}| + \cdots + |a_{n+l-1}|)(|a_{n-\kappa}| + \cdots + |a_{n-1}|) \to 0,$$

if $n \to +\infty$ for $\kappa = 1, 2, ...$ ∎

THEOREM 7.5. *Let $(a_n)_{n \geqslant 0}$ be a sequence in $\mathbb{D}$ satisfying the Máté–Nevai condition* (2.18) *for* $\kappa = 1, 2, ...$ *Then*

$$\lim_n \frac{1}{n+1} (|a_0| + |a_1| + \cdots + |a_n|) = 0 \tag{7.18}$$

Proof. Given ε, $\varepsilon > 0$ let $\Lambda(\varepsilon) = \{n: |a_n| \geqslant \varepsilon\}$. Since $(a_n)_{n \geqslant 0}$ satisfies (2.18) for $\kappa = 1, 2, ...,$ for every positive integer K there exists a positive integer $L = L(\varepsilon, K)$ such that

$$|a_{n+\kappa} a_n| < \varepsilon^2 \tag{7.19}$$

for $\kappa = 1, 2, ..., K$ and $n \geqslant L$.

Let $M(\varepsilon) = \Lambda(\varepsilon) \cap [L, +\infty)$. We claim that the sets

$$M(\varepsilon), M(\varepsilon) + 1, ..., M(\varepsilon) + K \tag{7.20}$$

do not intersect. Indeed, if $(M(\varepsilon) + j) \cap (M(\varepsilon) + i) \neq \varnothing$ for $i < j \leqslant K$, then there exists an integer n in $M(\varepsilon)$ such that $n + (j - i) \in M(\varepsilon)$. It follows that $\varepsilon^2 \leqslant |a_{n+(j-i)} a_n|$, which contradicts (7.19), since $1 \leqslant j - i \leqslant K$.

Now, let

$$d = d(\varepsilon) = \overline{\lim_n} \frac{\operatorname{Card} \Lambda(\varepsilon) \cap [0, n]}{n}$$

be the upper density of $\Lambda(\varepsilon)$. By (7.20) we have

$$\begin{aligned}
n &\geqslant L + \sum_{j=0}^{K} \operatorname{Card}(M(\varepsilon) + j) \cap [L, n] \\
&\geqslant L + (K+1) \operatorname{Card} M(\varepsilon) \cap [L, n] - K(K+1) \\
&= L + (K+1) \operatorname{Card} \Lambda(\varepsilon) \cap [L, n] - K(K+1) \\
&= L + (K+1) \operatorname{Card} \Lambda(\varepsilon) \cap [0, n] - K(K+1) - (K+1) \operatorname{Card} \Lambda(\varepsilon) \cap [0, L) \\
&\geqslant (K+1) \operatorname{Card} \Lambda(\varepsilon) \cap [0, n] - K(K+1) - K \cdot L.
\end{aligned}$$

Now we divide both parts of the above inequality by n and pass to the limit as $n \to +\infty$. It follows that

$$1 \geqslant (K+1) \cdot d.$$

Since K is an arbitrary positive integer, we obtain that $d = d(\varepsilon) = 0$. Since ε is an arbitrary positive number and since $(a_n)_{n \geqslant 0}$ is bounded, we obtain (7.18). ∎

Of course there are sequences which satisfy (7.18) but do not satisfy (2.18) for any κ. To obtain such examples we consider

$$\Lambda = \{2^n + \kappa : n = 0, 1, ..., \kappa = 0, 1, ..., n\}.$$

Since obviously

$$\operatorname{Card} \Lambda \cap [0, \mathcal{N}] \leqslant \sum_{2^n < \mathcal{N}} (n+1) \leqslant (\log \mathcal{N})^2,$$

the density of Λ is zero. On the other hand it is clear that

$$\operatorname{Card} \Lambda \cap (\Lambda + \kappa) = +\infty$$

for every κ. It follows that any sequence $(a_n)_{n \geqslant 0}$ such that $a_n = 0$ for $n \notin \Lambda$ and $0 < \delta < |a_n| < 1$ for $n \in \Lambda$ satisfies (7.18) but does not satisfy (2.18).

We conclude this section with a remark concerning Theorem 7.4. It is useful to compare the statement (8) of this theorem with Theorem 5.14. By (5.42) and (4.26) we obtain that

$$\begin{aligned}\frac{\Phi_{n+1}^*}{\Phi_n^*} &= \frac{k_n}{k_{n+1}} \cdot \frac{\varphi_{n+1}^*}{\varphi_n^*} \\ &= \frac{\omega_n}{\omega_{n-1}} \cdot \frac{1+o(1)}{1-zf} \cdot \prod_{\kappa=0}^{n-1} (1-\bar{\gamma}_\kappa f_\kappa) \cdot \frac{1-zf}{1+o(1)} \cdot \prod_{\kappa=0}^{n} \frac{1}{(1-\bar{\gamma}_\kappa f_\kappa)} \\ &= \frac{(1-|\gamma_n|^2)}{1-\bar{\gamma}_n f_n}(1+o(1)) = (1+z\bar{\gamma}_n f_{n+1})(1+o(1)) = 1+o(1)\end{aligned}$$

if and only if σ is a Rakhmanov measure.

8. CONVERGENCE OF CONTINUED FRACTIONS IN MEASURE

Recall that $E(\sigma) = \{\zeta \in \mathbb{T} : \sigma'(\zeta) > 0\}$ denotes the Lebesgue support of a probability measure σ. The following lemma shows that convergence of Wall's approximants on $E(\sigma)$ in the L^2-metric implies the convergence in $L^2(\mathbb{T})$.

LEMMA 8.1. *Let f be the Schur function of a probability measure σ such that*

$$\lim_n \int_{E(\sigma)} \left| f - \frac{A_n}{B_n} \right|^2 dm = 0. \tag{8.1}$$

Then

$$\lim_n \int_{\mathbb{T}} \left| f - \frac{A_n}{B_n} \right|^2 dm = 0. \tag{8.2}$$

Proof. It follows from (4.22) that f and A_n/B_n have matching Taylor polynomials of order n at $z=0$. Therefore, Parseval's identity and Cauchy's inequality imply

$$\int_{\mathbb{T}} \left| f - \frac{A_n}{B_n} \right|^2 dm = \int_{\mathbb{T}} |f|^2\, dm + \int_{\mathbb{T}} \left| \frac{A_n}{B_n} \right|^2 dm - 2\,\mathrm{Re} \int_{\mathbb{T}} f \cdot \frac{\overline{A}_n}{\overline{B}_n}\, dm$$
$$= \int_{\mathbb{T}} \left| \frac{A_n}{B_n} \right|^2 dm - \int_{\mathbb{T}} |f|^2\, dm + o(1). \tag{8.3}$$

Clearly,

$$\int_E \left| f - \frac{A_n}{B_n} \right|^2 dm \geqslant \left\{ \left(\int_E |f|^2\, dm \right)^{1/2} - \left(\int_E \left| \frac{A_n}{B_n} \right|^2 dm \right)^{1/2} \right\}^2,$$

which by (8.1) implies that

$$\int_E \left| \frac{A_n}{B_n} \right|^2 dm - \int_E |f|^2\, dm = o(1), \qquad n \to +\infty. \tag{8.4}$$

Since obviously $|f|=1$ on $\mathbb{T}\backslash E$, $|A_n/B_n|<1$ on $\mathbb{T}$ (see (4.16)), we obtain

$$\int_{\mathbb{T}\backslash E} \left| \frac{A_n}{B_n} \right|^2 dm - \int_{\mathbb{T}\backslash E} |f|^2\, dm < 0. \tag{8.5}$$

Taking the sum of (8.4) and (8.5), we obtain from (8.3) that

$$0 \leqslant \int_{\mathbb{T}} \left| f - \frac{A_n}{B_n} \right|^2 dm \leqslant o(1), \qquad n \to +\infty. \quad \blacksquare$$

Proof of Theorem 6. Let σ be a Rakhmanov measure. Then by Corollary 7.1 $f_n b_n \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$. Since the linear span of Poisson kernels is dense in $L^1(\mathbb{T})$ [12, Theorem 3.1] and $f_n b_n \in \mathscr{B}$, it follows that $\lim_n \int f_n b_n G\, dm = 0$ for every G in $L^1(\mathbb{T})$ (recall that $L^\infty(\mathbb{T})$ is the dual space of the Banach space $L^1(\mathbb{T})$ [48]); i.e., $f_n b_n \to 0$ in the $(*)$-weak topology of $L^\infty(\mathbb{T})$. Hence

$$\lim_n \int_E \zeta b_n f_n\, dm = 0 \tag{8.6}$$

for every measurable subset E of $\mathbb{T}$. Now, let

$$E = E(\sigma) = \{ \zeta \in \mathbb{T} : |f(\zeta)| < 1 \},$$

where f is the Schur function of σ; see (2.2). Integrating (6.4) over E and using (8.6), we obtain

$$\begin{aligned}\int_E |f_n|^2\,dm &= \int_E (1-g_n)\,dm + \operatorname{Re}\int_E \zeta b_n f_n\,dm \\ &\quad + \int_E (g_n-1)\operatorname{Re}(\zeta b_n f_n)\,dm \\ &\leqslant 2\int_E |1-g_n|\,dm + o(1), \qquad n\to+\infty. \end{aligned}\tag{8.7}$$

Resolving Eq. (1.3) with respect to zf_{n+1} (see also (4.25.3) and (4.25.4)), we obtain that

$$|f_{n+1}|\cdot\left|1-\frac{\bar{A}_n}{\bar{B}_n}f\right| = \left|f-\frac{A_n}{B_n}\right| \tag{8.8}$$

on $\mathbb{T}$. Taking into account that $A_n/B_n \in \mathscr{B}$, $f\in\mathscr{B}$, we obtain from (8.7) and (8.8) that

$$\int_E \left|f-\frac{A_n}{B_n}\right|^2 dm \leqslant 4\int_E |f_{n+1}|^2\,dm \leqslant 8\left(\int_E (1-g_n)^2\,dm\right)^{1/2} + o(1),$$

which implies (2.23) by Theorem 6.3 and Lemma 8.1. ∎

Proof of Theorem 5. If f is an inner function, then $|E(\sigma)|=0$ and we conclude that $\lim_n A_n/B_n = f$ in $L^2(\mathbb{T})$ by Lemma 8.1.

Now, let $\lim_n \gamma_n = 0$, γ_n being the Schur parameters of f. Then by Corollary 4.12 $f_n \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$. By Corollary 7.1 σ is a Rakhmanov measure, which implies that $L^2-\lim_n A_n/B_n = f$ by Theorem 6.

The necessity of the conditions of Theorem 5 follows from the lemma.

LEMMA 8.2. *Let* $f\in\mathscr{B}$ *and let* $|f|<1$ *on a set* E *of positive Lebesgue measure. If*

$$\lim_n \int_E \left|f-\frac{A_n}{B_n}\right| dm = 0, \tag{8.9}$$

then $\lim_n \gamma_n = 0$.

Proof. By (4.37) we have

$$\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \gamma_{n+1} z^{n+1} \frac{\omega_n}{B_n B_{n+1}}$$

$$= \frac{\gamma_{n+1} z^{n+1}}{\sqrt{1-|\gamma_{n+1}|^2}} \cdot \frac{\sqrt{\omega_n}}{B_n} \cdot \frac{\sqrt{\omega_{n+1}}}{B_{n+1}}. \tag{8.10}$$

Using (4.15), we obtain from (8.10) that

$$\int_E \left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right| dm = \frac{|\gamma_{n+1}|}{\sqrt{1-|\gamma_{n+1}|^2}} \cdot \int_E \left(1 - \left|\frac{A_n}{B_n}\right|^2\right)^{1/2} \times \left(1 - \left|\frac{A_{n+1}}{B_{n+1}}\right|^2\right)^{1/2} dm. \tag{8.11}$$

It follows from (8.9) that $A_n/B_n \rightrightarrows f$ on E. Therefore

$$\lim_n \int_E \left(1 - \left|\frac{A_n}{B_n}\right|^2\right)^{1/2} \left(1 - \left|\frac{A_{n+1}}{B_{n+1}}\right|^2\right)^{1/2} dm = \int_E (1-|f|^2)\, dm > 0$$

by Lebesgue's dominated convergence theorem [53, Chap. VII, Sect. 3, Theorem VII.3.1]. On the other hand,

$$\lim_n \int_E \left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right| dm = 0$$

by (8.9). It follows that $\lim_n |\gamma_{n+1}| \, (1-|\gamma_{n+1}|^2)^{-1/2} = 0$. ▮

COROLLARY 8.3. *Let σ be a probability measure with Schur-function f and let $|E(\sigma)| > 0$. Then σ is in Nevai's class if and only if*

$$\lim_n \int_{E(\sigma)} |f_n|^2 \, dm = 0. \tag{8.12}$$

Proof. By the Khinchin–Ostrovskii's theorem [44], (8.12) implies that $f_n \rightrightarrows 0$ in $\mathbb{D}$. It follows that $\lim_n \gamma_n = \lim_n f_n(0) = 0$. On the other hand, if $\lim_n \gamma_n = 0$, then $f_n \rightrightarrows 0$ in $\mathbb{D}$ by Corollary 4.12. By Corollary 7.1 σ is a Rakhmanov measure. Now (8.12) follows from (8.7). ▮

Remark. Recall (see Section 2) that in [59] Totik constructed examples of measures σ in Nevai's class such that $0 < |E(\sigma)| < \varepsilon$, where ε can be arbitrary small.

COROLLARY 8.4. *Let σ be a probability measure with $|E(\sigma)|>0$ which does not belong to Nevai's class. Then the sequence of the Wall approximants $(A_n/B_n)_{n\geqslant 0}$ diverges in measure on any subset of positive Lebesgue measure in $E(\sigma)$.*

Proof. By Wall's Theorem $A_n B_n \rightrightarrows f$ uniformly on compact subsets of $\mathbb{D}$. Since the linear span of Poisson kernels is dense in $L^1(\mathbb{T})$ [12, Theorem 3.1] and $A_n/B_n \in \mathscr{B}$, it follows that $(*)-\lim_n A_n/B_n = f$ in the $(*)$-weak topology of $L^\infty(\mathbb{T})$. Therefore

$$\lim_n \int_{\mathbb{T}} (A_n/B_n)\, h\, dm = \int_{\mathbb{T}} hf\, dm \tag{8.13}$$

for any h, $h \in L^1(\mathbb{T})$. Now, suppose that $A_n/B_n \Rightarrow g$ on $E \subset E(\sigma)$, $|E|>0$. Then by Lebesgue's dominated convergence theorem [53, Chap. VII, Sect. 3, Theorem VII.3.1] and (8.13) we obtain that

$$\int_{\mathbb{T}} gh\, dm = \int_{\mathbb{T}} fh\, dm$$

for every h supported by E, which implies that $g=f$ a.e. on E. Applying again Lebesgue's theorem, we obtain (8.9), which by Lemma 8.2 implies that σ is in Nevai's class. ∎

Now we prove Nevai's results (2.3) stated in Section 2. We summarize some useful inequalities of Section 6 in the following lemma.

LEMMA 8.5. *Let σ be a probability measure on $\mathbb{T}$ with Geronimus parameters $(a_n)_{n\geqslant 0}$ and with Schur function f, $(\varphi_n)_{n\geqslant 0}$ the orthogonal polynomials in $L^2(d\sigma)$. Then*

$$\tfrac{1}{2}|a_n|^2 \leqslant \tfrac{1}{2}\int_{\mathbb{T}} |f_n|^2\, dm \leqslant \int_{\mathbb{T}} |1-|\varphi_n|^2 \sigma'|\, dm \leqslant 12\cdot \int_{\mathbb{T}} |f_n|\, dm. \tag{8.14}$$

Proof. By Geronimus' theorem $a_n = f_n(0)$. Therefore the first inequality in (8.14) follows by the mean-value theorem and by Cauchy's inequality. The second inequality follows from (6.6) and (6.7). The third inequality coincides with (6.25). ∎

It follows from (5.11) that A_{n+l}/B_{n+l} is the Schur function of the probability measure $|\varphi_{n+l+1}|^{-2}\, dm$, $l=0, 1, \ldots$. We denote by f_n^l the Schur function of order n for A_{n+l}/B_{n+l}. By (4.17)

$$a_n = f_n^l(0), \qquad f_n^0 \equiv a_n, \qquad n=0, 1, \ldots, \quad l=0, 1, \ldots. \tag{8.15}$$

We begin with the second equivalence (2.3).

THEOREM 8.6 (Nevai [41]). *Let σ be a probability measure on $\mathbb{T}$ and $(\varphi_n)_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then σ is in Nevai's class if and only if*

$$\lim_n \inf_{l\geqslant 0} \int_{\mathbb{T}} \left| \frac{|\varphi_n|^2}{|\varphi_{n+l+1}|^2} - 1 \right| dm = 0. \tag{8.16}$$

Proof. This is immediate from (8.14) if we put $\sigma' = |\varphi_{n+l+1}|^{-2}$. Indeed, by (8.15) $a_n = f_n^l(0)$ for every l, $l = 0, 1, \ldots$, and $\int_{\mathbb{T}} |f_n^0| \, dm = |a_n|$. ∎

The proof of the first equivalence (2.3) is more complicated.

THEOREM 8.7 (Nevai [41]). *Let σ be a probability measure on $\mathbb{T}$ and $(\varphi_n)_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then σ is an Erdős measure if and only if*

$$\lim_n \sup_{l\geqslant 0} \int_{\mathbb{T}} \left| \frac{|\varphi_n|^2}{|\varphi_{n+l+1}|^2} - 1 \right| dm = 0 \tag{8.17}$$

Proof. By (1.3), see also (4.25.3) and (4.25.4), we have

$$\begin{aligned} zf_n^l &= \frac{B_{n-1}(A_{n+l}/B_{n+l}) - A_{n-1}}{B_{n-1}^* - A_{n-1}^*(A_{n+l}/B_{n+l})} \\ &= \frac{(A_{n+l}/B_{n+l}) - (A_{n-1}/B_{n-1})}{(B_{n-1}^*/B_{n-1}) - (A_{n-1}^*/B_{n-1})(A_{n+l}/B_{n+l})}. \end{aligned} \tag{8.18}$$

Suppose first that (8.17) holds. Then $L^1 - \lim_n |\varphi_n/\varphi_{n+1}|^2 = 1$ and therefore σ is a Rakhmanov measure by (1) $\Leftrightarrow$ (6) of Theorem 7.4. By Theorem 6 $A_n/B_n \rightrightarrows f$ on $\mathbb{T}$. Passing to the limit in (8.18) as $l \to +\infty$, we obtain that $f_n^l \rightrightarrows f_n$, $l \to +\infty$.

Now let $\sigma' = |\varphi_{n+l+1}|^{-2}$ in (8.14). Applying Lebesgue's dominated convergence theorem, we obtain from the second inequality (8.14) that

$$\int_{\mathbb{T}} |f_n|^2 \, dm \leqslant 2 \sup_{l\geqslant 0} \int_{\mathbb{T}} \left| \frac{|\varphi_n|^2}{|\varphi_{n+l+1}|^2} - 1 \right| dm,$$

which implies that σ is an Erdős measure by Theorem 1.

Now let σ be an Erdős measure. Then $\lim_n a_n = 0$ by Rakhmanov's theorem (see Corollary 2.3). It follows that σ is a Rakhmanov measure by (1) $\Leftrightarrow$ (4) of Theorem 7.4. By Theorem 6 $A_n/B_n \rightrightarrows f$ on $\mathbb{T}$.

Multiplying both sides of (8.18) by the denominator of the second fraction in (8.18) and using the triangle inequality, we obtain that

$$\int_{\mathbb{T}} |f_n^l| \, (1 - |f|) \, dm \leqslant \int_{\mathbb{T}} \left| \frac{A_{n+l}}{B_{n+l}} - \frac{A_{n-1}}{B_{n-1}} \right| dm + \int_{\mathbb{T}} \left| \frac{A_{n-1}}{B_{n-1}} - f \right| dm.$$

It follows that

$$\lim_n \sup_l \int_{\mathbb{T}} |f_n^l|\,(1-|f|)\,dm = 0.$$

Therefore

$$\lim_n \sup_l \int_E |f_n^l|\,dm = 0 \tag{8.19}$$

for any measurable set E, $E \subset \mathbb{T}$, with $\sup_E |f| < 1$. Since $|f| < 1$ a.e. on $\mathbb{T}$, for every $\varepsilon > 0$ there is E with $\sup_E |f| < 1$ such that $|\mathbb{T} \setminus E| < \varepsilon$. Observing that $f_n^l \in \mathscr{B}$, we obtain from (8.19) that

$$\lim_n \sup_l \int_{\mathbb{T}} |f_n^l|\,dm = 0.$$

The result now follows from the third inequality (8.14) with $\sigma' = |\varphi_{n+l+1}|^{-2}$. ∎

The following corollary is immediate from (8.14) and Theorem 8.7.

Corollary 8.8. *A probability measure σ is an Erdős measure if and only if*

$$\lim_n \sup_l \int_{\mathbb{T}} |f_n^l|^2\,dm = 0. \tag{8.20}$$

Now we show how Theorem 7 can be obtained from Theorem 5.

Proof of Theorem 7. We observe that by (5.5)

$$\frac{\psi_{n+1}^*(z)}{\varphi_{n+1}^*(z)} = \frac{1 + z(A_n/B_n)}{1 - z(A_n/B_n)}. \tag{8.21}$$

Clearly, (8.21) shows that $\psi_{n+1}^*/\varphi_{n+1}^*$ is the $(n+1)$th approximant of the continued fraction (1.16).

For every z, $z \in \mathbb{T}$,

$$\tau_z(x) = \frac{1 + zw}{1 - zw} = -\bar{z}\,\frac{1 + zw}{w - \bar{z}}$$

is a superposition of two rotations of the Riemann sphere, which keep invariant the metric $k(w_1, w_2)$; see (3.14).

It follows that

$$k\left(\frac{\psi_{n+1}^*}{\varphi_{n+1}^*}, F_\sigma\right) = k\left(\frac{A_n}{B_n}, f\right) \tag{8.22}$$

a.e. on $\mathbb{T}$. Let η_n be the function on $\mathbb{T}$ defined by the left-hand side of (8.22). Since $A_n/B_n \in \mathscr{B}$, $f \in \mathscr{B}$, we obtain from (8.22) that

$$\frac{1}{2}\left|f - \frac{A_n}{B_n}\right| \leqslant \eta_n \leqslant \left|f - \frac{A_n}{B_n}\right|.$$

Now we complete the proof by Theorem 5 and by the observation that $\eta_n \Rightarrow 0$ on $\mathbb{T}$ if and only if $\psi_n^*/\varphi_n^* \Rightarrow F_\sigma$ on $\mathbb{T}$. ∎

Theorem 8 is an easy corollary of Theorem 7.

Proof of Theorem 8. Since $\operatorname{Re} \psi_n^*/\varphi_n^* > 0$ in $\mathbb{D}$, see (5.10), we obtain that

$$\int_{\mathbb{T}} \left|\frac{\psi_n^*}{\varphi_n^*}\right|^s dm \leqslant \frac{1}{\cos(\pi s/2)}, \qquad 0 < s < 1, \tag{8.23}$$

by Smirnov's theorem [12, Chap. III, Sect. 2, Theorem 2.4].

Given p, $0 < p < 1$, we fix any $r > 1$ with $s = rp < 1$. Then for every e, $e \subset \mathbb{T}$, we have by Hölder's inequality

$$\int_e \left|\frac{\psi_n^*}{\varphi_n^*}\right|^p dm \leqslant |e|^{1/r'} \cdot \left(\int_e \left|\frac{\psi_n^*}{\varphi_n^*}\right|^{pr} dm\right)^{1/r} \leqslant \frac{|e|^{1-1/r}}{(\cos(\pi rp/2))^{1/r}}. \tag{8.24}$$

If σ is a singular measure or σ is in Nevai's class, then $\psi_n^*/\varphi_n^* \Rightarrow F_\sigma$ by Theorem 7. For every ε, $\varepsilon > 0$, we put

$$e(\varepsilon, n) = \{\zeta \in \mathbb{T} : |(\psi_n^*/\varphi_n^*) - F_\sigma| > \varepsilon\}.$$

It follows that

$$\lim_{n \to +\infty} |e(\varepsilon, n)| = 0. \tag{8.25}$$

Now we have

$$\overline{\lim_n} \int_{\mathbb{T}} \left|\frac{\psi_n^*}{\varphi_n^*} - F_\sigma\right|^p dm \leqslant \varepsilon^p + \overline{\lim_n} \int_{e(\varepsilon, n)} \left|\frac{\psi_n^*}{\varphi_n^*}\right|^p + \overline{\lim_n} \int_{e(\varepsilon, n)} |F_\sigma|^p\, dm$$
$$\leqslant \varepsilon^p + \frac{2\, \overline{\lim}_n\, |e(\varepsilon, n)|^{1-1/r}}{(\cos(\pi rp/2))^{1/r}} = \varepsilon^p,$$

by (8.24) and (8.25) and by Lebesgue's dominated convergence theorem. Since ε is arbitrary, we obtain (2.25). ▮

We consider now one more corollary of Theorem 7.

COROLLARY 8.9. *Let σ be either a singular measure or a measure in Nevai's class. Then for every p, $0<p<+\infty$,*

$$\lim_n \int_{\mathbb{T}} \left|\log \frac{\psi_n^*}{\varphi_n^*} - \log F_\sigma\right|^p dm = 0. \tag{8.26}$$

Proof. By (5.9) there exists a continuos function $A_n(\zeta)$ on $\mathbb{T}$, such that $\|A_n\|_\infty < \pi/2$ and $\operatorname{Arg}(\psi_n^*/\varphi_n^*) = A_n$ on $\mathbb{T}$. We have

$$\log \frac{\psi_n^*}{\varphi_n^*} = \log \left|\frac{\psi_n^*}{\varphi_n^*}\right| + iA_n. \tag{8.27}$$

It follows that $\log |\psi_n^*/\varphi_n^*|$ is the harmonic conjugate of $-A_n$. By Theorem 7 $\psi_n^*/\varphi_n^* \Rightarrow F_\sigma$ and therefore $A_n \Rightarrow A = \operatorname{Arg} F_\sigma$. Since $(A_n)_{n\geqslant 0}$ is uniformly bounded, we obtain (8.26) for every p, $1<p<+\infty$, by Lebesgue's dominated convergence theorem and by Riesz' theorem [12, Chap. III, Theorem 2.3]. For $p, 0<p\leqslant 1$, the result follows by Hölder inequality. ▮

It is interesting to compare (8.26) with (2.13). Recall that σ_{-1} is the probability measure with Geronimus parameters $(-a_n)_{n\geqslant 0}$, where $(a_n)_{n\geqslant 0}$ are the Geronimus parameters of σ, see Section 1. Clearly, $-f$ is the Schur function of σ_{-1}, see (1.3). It follows that $F_{\sigma_{-1}} = 1/F_\sigma$ and therefore

$$\sigma'_{-1} = \frac{\sigma'}{|F_\sigma|^2} \qquad \text{a.e. on } \mathbb{T}. \tag{8.28}$$

Applying Theorem 2.5 separately to σ and σ_{-1}, we obviously obtain that

$$\lim_n \int_{\mathbb{T}} \left|\log \frac{|\psi_n|}{|\varphi_n|} - \log |F_\sigma|\right| dm = 0,$$

if σ is a Szegő measure (which, in view of (8.28), is equivalent to σ_{-1} being a Szegő measure). Corollary 8.9 says that although for singular measures and for measures of Nevai's class we cannot guarantee the convergence of $\log |\varphi_n^*|$ in the L^1-metric, as we can for Szegő measures, still we can guarantee more than that for $\log |\psi_n^*/\varphi_n^*|$.

The following theorem extends Corollary 5.11 to Nevai's class.

THEOREM 8.10. *Let σ be either a singular measure or a measure in Nevai's class. Then for every p, $0<p<1$,*

$$\lim_n \int_{\mathbb{T}} \left| \frac{1}{|\varphi_n|^2} - \sigma' \right|^p dm = 0. \tag{8.29}$$

Proof. We suppose first that $p<1/4$. Then, by (5.11) and Cauchy's inequality, we have

$$\begin{aligned} \int_{\mathbb{T}} \left| \frac{1}{|\varphi_{n+1}|^2} - \sigma' \right|^p dm \leqslant & \left(\int_{\mathbb{T}} \left| 1 - \left| \frac{A_n}{B_n} \right|^2 - \sigma' \left| 1 - z \frac{A_n}{B_n} \right|^2 \right|^{2p} dm \right)^{1/2} \\ & \times \left(\int_{\mathbb{T}} \frac{dm}{|1 - z(A_n/B_n)|^{4p}} \right)^{1/2}. \end{aligned} \tag{8.30}$$

The second integral on the right-hand side of (8.30) is uniformly bounded, by Smirnov's theorem, since $4p<1$ and $\operatorname{Re}(1-z(A_n/B_n))>0$ in $\mathbb{D}$. The first integral on the right-hand side of (8.30) tends to zero as $n \to +\infty$ by Lebesgue's dominated convergence theorem, since by Theorem 5

$$1 - \left| \frac{A_n}{B_n} \right|^2 - \sigma' \left| 1 - z \frac{A_n}{B_n} \right|^2 \Rightarrow 1 - |f|^2 - \sigma' \, |1 - zf| = 0;$$

see (2.2).

The proof can be completed now by use of convexity arguments. Let

$$\delta_n(p) = \int_{\mathbb{T}} \left| \frac{1}{|\varphi_n|^2} - \sigma' \right|^p dm, \qquad 0<p\leqslant 1.$$

Clearly, $\delta_n(1) \leqslant 2$. The function δ_n is logarithmic convex [56, Theorem 10.12]. Now, let $p<1$. We pick any $p_0 < \min(1/4, p)$. Then $p = t_0 p_0 + t_1$, where $t_0 + t_1 = 1$, $t_i > 0$. By the logarithmic convexity of δ_n we have

$$\delta_n(p) \leqslant \delta_n(p_0)^{t_0} \, \delta_n(1)^{t_1} \leqslant 2^{t_1} \cdot \delta_n(p_0)^{t_0} \to 0,$$

since $p_0 < 1/4$. ∎

The following corollary is immediate from Theorem 8.10.

COROLLARY 8.11. *Let σ be a measure in Nevai's class. Then*

$$\frac{1}{|\varphi_n|^2} \Rightarrow \sigma', \qquad \text{on } \mathbb{T}, \tag{8.31}$$

$$|\varphi_n|^2 \, \sigma' \Rightarrow \mathbb{1}_{E(\sigma)}. \tag{8.32}$$

THEOREM 8.12. *Let σ be a measure in Nevai's class. Then for every p,* $0<p<1$,

$$\lim_n \int_{\mathbb{T}} ||\varphi_n|^2 \sigma' - \mathbb{1}_{E(\sigma)}|^p \, dm = 0 \tag{8.33}$$

Proof. It is similar to that of Theorem 8.10. If $p<1/4$, then by (2.12) and by Cauchy's inequality

$$\int_{\mathbb{T}} ||\varphi_n|^2 \sigma' - \mathbb{1}_{E(\sigma)}|^p \, dm \leqslant \left(\int_{E(\sigma)} |1-|f_n|^2 - |1-\zeta b_n f_n|^2|^{2p} \, dm \right)^{1/2} \times \left(\int_{\mathbb{T}} \frac{dm}{|1-\zeta b_n f_n|^{4p}} \right)^{1/2}. \tag{8.34}$$

By Smirnov's theorem and by Corollary 8.3 the right-hand side of (8.34) tends to zero as $n \to +\infty$. For $1/4 \leqslant p < 1$ we apply convexity arguments. ∎

COROLLARY 8.13. *Let σ be a measure in Nevai's class. Then for every p,* $0<p<1$,

$$\lim_n \int_{\mathbb{T}} (|\varphi_n|^2 \sigma')^p \, dm = |E(\sigma)|.$$

Proof. This follows from (8.33) by the elementary inequality $|a^p - b^p| \leqslant |a-b|^p$, $0<p<1$. ∎

9. INNER FUNCTIONS AND UNIMODULAR FUNCTIONS ON AN ARC

Recall that a function f in $\mathscr{B}$ is called an inner function if $|f|=1$ a.e. on $\mathbb{T}$ [12, Chap. II, Sect. 6, 22]. By (2.2) inner functions are Schur functions of singular measures. Therefore it follows from Szegő's theorem, see Corollary 5.12, that

$$\sum_{n=0}^{\infty} |\gamma_n|^2 = +\infty \tag{9.1}$$

for the Schur parameters of any inner function. This, however, can be shown directly. Indeed, given an inner function f we obtain from (4.25.3) that

$$\int_{\mathbb{T}} \log |B_n f - A_n| \, dm = \sum_{\kappa=0}^{n} \int_{\mathbb{T}} \log |1 - \bar{\gamma}_\kappa f_\kappa| \, dm,$$

since f_{n+1} is an inner function; see (6.1). Observing that $B_n(0) = 1$, we obtain by the mean-value theorem that

$$\int_{\mathbb{T}} \log \left| f - \frac{A_n}{B_n} \right| dm = \sum_{\kappa=0}^{n} \log(1 - |\gamma_\kappa|^2). \tag{9.2}$$

By Lemma 8.1 $A_n/B_n \Rightarrow f$, if f is an inner function, which yields (9.1).

On the other hand, $p = 2$ is the largest value such that (9.1) takes place for all inner functions. For example, it is shown in [28] that there exist infinite Blaschke products such that

$$\sum_{n=0}^{\infty} |\gamma_n|^p < +\infty$$

for every p, $p > 2$. Another example is provided by Theorem 5.

Corollary 9.1. *Suppose that the Schur parameters $(\gamma_n)_{n \geqslant 0}$ of f, $f \in \mathscr{B}$, satisfy the Máté–Nevai condition*

$$\lim_n \gamma_n \gamma_{n+\kappa} = 0$$

for $\kappa = 1, 2, \ldots$, but $\overline{\lim}_n \gamma_n > 0$. Then f is an inner function.

Proof. Let σ be the probability measure with Schur function f. By Geronimus' theorem the Geronimus parameters of σ satisfy the Máté–Nevai condition for $\kappa = 1, 2, \ldots$, which implies that σ is a Rakhmanov measure by Theorem 4. By Theorem 6 Wall's approximants A_n/B_n converge to f in $L^2(\mathbb{T})$. Since $\overline{\lim}_n |\gamma_n| > 0$, it follows from Theorem 5 that f is an inner function. ∎

Corollary 9.2. *Let $(n_\kappa)_{\kappa \geqslant 0}$ be a sequence of nonnegative integers such that*

$$\lim_\kappa n_{\kappa+1} - n_\kappa = +\infty \tag{9.3.1}$$

and let $\Lambda = \{n_\kappa : \kappa = 0, 1, ...\}$. Suppose that the Schur parameters $(\gamma_n)_{n \geqslant 0}$ of f, $f \in \mathscr{B}$, satisfy

$$\lim_{n \notin \Lambda} |\gamma_n| = 0, \tag{9.3.2}$$

$$\overline{\lim_{n \in \Lambda}} |\gamma_n| > 0. \tag{9.3.3}$$

Then f is an inner function.

Proof. In view of (9.3.1) n and $n+\kappa$ cannot both belong to Λ for infinitely many n's for a fixed positive κ. It follows by (9.3.2) that the sequence $(\gamma_n)_{n \geqslant 0}$ satisfies the Máté–Nevai condition for $\kappa = 1, 2, ...$ By Corollary 9.1 we conclude—see (9.3.3)—that f is an inner function. ∎

Given any Szegő measure with Schur function f we can construct infinitely many inner functions with Schur parameters which are "close" to the Schur parameters of f. Indeed, we can take any gap subset Λ satisfying (9.3.1) and redefine the parameters γ_n for $n \in \Lambda$ to satisfy (9.3.3). Then by Corollary 9.2 the functions obtained are inner.

Our next goal is to characterize inner functions in terms of Schur functions. The key is the following theorem, which allows one to extract some information on the behavior of Schur functions for general probability measures.

THEOREM 9.3. *Let σ be a probability measure on $\mathbb{T}$ with Schur functions $(f_n)_{n \geqslant 0}$. Then for every p, $p < 1/4$, there exists a constant c_p, $c_p > 0$, such that*

$$\left(\int_E (\sigma')^p \, dm\right)^{1/p} \leqslant c_p \cdot \int_E (1 - |f_n|^2) \, dm \tag{9.4}$$

for any measurable subset E of $\mathbb{T}$.

Proof. Applying (2.12) and Hölder's inequality with $1/p$ and $1/(1-p)$, we obtain that

$$\begin{aligned}
&\int_E (\sigma')^p \, dm \\
&\quad = \int_E (1 - |f_n|^2)^p \cdot \frac{dm}{|\varphi_n|^{2p} \, |1 - \zeta b_n f_n|^{2p}} \qquad (9.5)\\
&\quad \leqslant \left(\int_E (1 - |f_n|^2) \, dm\right)^p \cdot \left(\int_E \frac{dm}{\{|\varphi_n| \, |1 - \zeta b_n f_n|^{2p/(1-p)}\}}\right)^{1-p}.
\end{aligned}$$

Since $(1-p)/p>1$, we can apply Hölder's inequality with $(1-p)/p$ and $(1-p)/(1-2p)$ to the second integral on the right-hand side of (9.5):

$$\int_E \frac{dm}{|\varphi_n|^{2p/(1-p)}\,|1-\zeta b_n f_n|^{2p/(1-p)}} \leqslant \left(\int_E \frac{dm}{|\varphi_n|^2}\right)^{p/(1-p)} \left(\int_E \frac{dm}{|1-\zeta b_n f_n|^{2p/(1-2p)}}\right)^{(1-2p)/(1-p)}. \tag{9.6}$$

Since $4p<1$, the second integral in the right-hand side of (9.6) is bounded by a constant by Smirnov's theorem [12, Chap. III, Section 2, Theorem 2.4], while the first is bounded by 1, since $|\varphi_n|^{-2}\,dm$ is a probability measure (put $z=0$ in (5.11)). Combining (9.5) and (9.6), we obtain (9.4). ∎

The meaning of (9.4) can be summarized as follows: if a measurable set E carries a positive mass of the absolutely continuous part of σ, then the corresponding Schur functions satisfy

$$\overline{\lim_n} \int_E |f_n|^2\,dm \leqslant |E| - c_p^{-1}\left(\int_E (\sigma')^p\,dm\right)^{1/p} < |E|. \tag{9.7}$$

Corollary 9.4. *Let $f\in\mathscr{B}$. Then f is an inner function if and only if*

$$\overline{\lim_n} \int_{\mathbb{T}} |f_n|^2\,dm = 1. \tag{9.8}$$

Proof. Let $E=\mathbb{T}$ in (9.4). Then it follows from (9.8) that the left-hand side of (9.4) is zero, which implies that $\sigma'=0$ a.e. on $\mathbb{T}$. On the other hand if f is an inner function, then $|f_n|=1$ a.e. on $\mathbb{T}$ for every n; see (6.1). ∎

Corollary 9.5 [46, Lemma 4]. *Let $f\in\mathscr{B}$ and let*

$$\overline{\lim}_n\, |\gamma_n| = 1. \tag{9.9}$$

Then f is an inner function.

Proof. By the mean-value property of f_n we have

$$|\gamma_n| = |f_n(0)| = \left|\int_{\mathbb{T}} f_n\,dm\right| \leqslant \left(\int_{\mathbb{T}} |f_n|^2\,dm\right)^{1/2}. \quad ∎$$

We consider now inner functions satisfying (2.34).

Proof of Theorem 10. Let F be the closed set of the limit points of the sequence $-\bar{\gamma}_n\gamma_{n-1}$, $n=1,2,\ldots$. By (2.34) $F\subset\mathbb{T}$. By Theorem 3, $f_n b_n$ is the

Schur function of the probability measure $|\varphi_n|^2\,d\sigma$. Next, $f_n(0)\,b_n(0) = -\gamma_n\bar{\gamma}_{n-1}$. The family $f_n b_n$ is compact and its limit points in $\mathscr{B}$ are constant functions with values in the complex conjugate set to F. It follows that the set of all weak-(*) limit points of the family $|\varphi_n|^2\,d\sigma$, $n=0, 1, ...,$ is exactly the set $\{\delta_\tau : \tau \in F\}$. This obviously implies that $F \subset \operatorname{supp}(\sigma)$. Moreover, F is contained in the derived set of $\operatorname{supp}(\sigma)$. Indeed, suppose to the contrary that τ, $\tau \in F$, is an isolated point of $\operatorname{supp}(\sigma)$. Then there exists a subset Λ of the set of all positive integers such that

$$\lim_{n \in \Lambda} \int_{\mathbb{T}} h\,|\varphi_n|^2\,d\sigma = h(\tau) \tag{9.10}$$

for every continuous function h. On the other hand

$$\sum_{n=0}^{\infty} |\varphi_n(\tau)|^2 = \sigma\{\tau\}^{-1}; \tag{9.11}$$

see [1, Theorem 20.2] or the remark on p. 405 of [18] for an elementary proof of (9.11) with the inequality $\leqslant$ instead of the equality. Clearly, (9.11) implies that

$$\lim_n |\varphi_n(\tau)| = 0,$$

which, however, contradicts (9.10) if $\operatorname{supp}(h) \cap \operatorname{supp}(\sigma) = \tau$ and $h(\tau) = 1$.

To prove that the derived set of $\operatorname{supp}(\sigma)$ is contained in F we apply Worpitsky's theorem; see Section 2, (2.30).

By equivalence transforms we can replace (1.7) with

$$\begin{gathered}
\mathop{\mathsf{K}}_{n=1}^{\infty} (a_n(z)/1), \qquad a_1 \equiv \gamma_0, \qquad a_2 = -\frac{(1-|\gamma_0|^2)(\gamma_1/\gamma_0)\,z}{1+(\gamma_1/\gamma_0)\,z}, \\
a_n(z) = -\frac{(1-|\gamma_{n-2}|^2)(\gamma_{n-1}/\gamma_{n-2})\,z}{(1+(\gamma_{n-1}/\gamma_{n-2})\,z)(1+(\gamma_{n-2}/\gamma_{n-3})\,z)}, \qquad n = 3, 4, ...
\end{gathered} \tag{9.12}$$

Let I be any closed arc in $\mathbb{T}\backslash F$. Then it is clear from (9.12) that the denominators of $a_n(z)$ in (9.12) are uniformly bounded away from zero in an open neighborhood V of I. It follows from (2.34) that

$$\lim_n \sup_{z \in V} |a_n(z)| = 0.$$

Then by Worpitsky's theorem $\mathsf{K}_{n=\mathcal{N}}^{\infty}(a_n(z)/1)$ converges absolutely and uniformly in V (see Section 3) to a holomorphic function if $\mathcal{N}$ is sufficiently large. Since by Wall's theorem the continued fraction (1.7) converges absolutely and uniformly on compact subsets of $\mathbb{D}$ to the Schur function f of σ, we obtain that f admits a meromorphic extension to V. Since $f \in \mathscr{B}$ we conclude that f is holomorphic on I. Clearly, possible points of $\operatorname{supp}(\sigma)$ in I are located in the zeros of the holomorphic function $1-zf$. It follows that $\operatorname{supp}(\sigma) \cap (\mathbb{T} \backslash F)$ consists only of isolated points. ▮

By Nevanlinna's factorization theorem [12, Chap. II, Theorem 5.4] any inner function f can be represented as

$$f = \lambda B \cdot S, \tag{9.13}$$

where $\lambda \in \mathbb{T}$, $|\lambda| = 1$, B is the Blaschke product constructed from the zeros of f in $\mathbb{D}$,

$$B(z) = z^{n_f} \cdot \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \cdot \frac{z_n - z}{1 - \bar{z}_n z}, \qquad n_f \geqslant 0,$$

and S is a singular inner function

$$S(z) = \exp\left\{ -\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu \right\},$$

where μ is a singular measure.

By Theorem 2.7 and (9.13) we obtain the following characterization of the Schur functions of probability measures with a one-point derived set.

Corollary 9.6. *Let σ be a probability measure on $\mathbb{T}$ with infinite support. Then the following statements are equivalent*:

(1) *the derived set of* $\operatorname{supp}(\sigma)$ *is* $\{\tau\}$;

(2) $d\mu = a \cdot \delta_\tau$, $a \geqslant 0$, τ *is the only limit point of* $\{z_n\}$.

Let us consider now $f = S$ with $d\mu = \delta_1$. Since f is real on $(-1, 1)$, we conclude that the Schur parameters γ_n of f are also real. By Theorem 2.7 $\lim_n \gamma_n \gamma_{n-1} = -1$, which shows that $\{\gamma_n\}$ has two limit points $\{+1\}$, $\{-1\}$.

Proof of Theorem 9. Let f be the Schur function of σ. It follows from (1.3) that the Schur parameters of $f^\theta(z) = f(e^{i\theta} z)$ are given by the sequence $\gamma_0, e^{i\theta}\gamma_1, ..., e^{in\theta}\gamma_n, ...$ Hence we may suppose without loss of generality that in (2.32.2) $\theta = 0$.

We apply Pringsheim's theorem to the continued fraction

$$K_{\mathcal{N}}(z) = \mathop{\mathrm{K}}_{n=\mathcal{N}}^{\infty} \left(-\frac{(1 - |\gamma_{n-1}|^2)(\gamma_n/\gamma_{n-1})\, z}{1 + (\gamma_n/\gamma_{n-1})\, z} \right).$$

Then (2.31) takes the following form

$$\left|\frac{\gamma_{n-1}}{\gamma_n}+z\right| \geqslant (1-|\gamma_{n-1}|^2)\,|z| + \left|\frac{\gamma_{n-1}}{\gamma_n}\right|, \qquad n=1, 2, ... \tag{9.14}$$

Let Δ_n be the open disc centered at $c_n = -\gamma_{n-1}/\gamma_n$ with radius $(1-|\gamma_{n-1}|^2) + |\gamma_{n-1}/\gamma_n|$. By (2.32.1) we have

$$0 < \inf_n |c_n| \leqslant \sup_n |c_n| < +\infty.$$

It follows from (2.32.2) (with $\theta = 0$) that for all sufficiently large n the centers c_n lie in an arbitrarily small angel with vertex at $z = 0$ and bisectrix directed along the negative real semi-axis. Since $\sup_n (1-|\gamma_{n-1}|^2) < 1$, we can find an open neighborhood U of the point $z = 1$ such that $\Delta_n \cap U = \varnothing$ for $n = \mathcal{N}, \mathcal{N}+1, ...$, where $\mathcal{N}$ is a large positive integer. Squeezing U if necessary, we obtain that (9.14) holds in U for $n \geqslant \mathcal{N}$. By Pringsheim's theorem this implies that the continued fraction $K_{\mathcal{N}}$ converges to a bounded holomorphic function in U. It follows that

$$f(z) = \gamma_0 (1 + K_1(z))^{-1}$$

is meromorphic in U. Since f is bounded in $\mathbb{D} \cap U$, we conclude that f is holomorphic on some open arc I centered at $z = 1$, and that the continued fraction (1.7) converges to f uniformly on I:

$$\lim_n \sup_I \left| f - \frac{A_n}{B_n} \right| = 0.$$

Since $\underline{\lim}_n |\gamma_n| > 0$, we obtain by Lemma 8.2 that $|f| = 1$ on I. By (1.14) $\operatorname{supp}(\sigma) \cap I$ consists of the roots of the holomorphic function $1 - zf$ on I. ∎

10. SCHUR FUNCTIONS OF SMOOTH MEASURES

We derive Theorem 11 from the following theorem.

THEOREM 10.1. *Let σ be a probability measure with Schur functions $(f_n)_{n \geqslant 0}$ and let $(\varphi_n)_{n \geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Suppose that $\|f_n\|_\infty < 1/2$. Then*

$$\big| |\varphi_n|^2 \sigma' - 1 \big| < \frac{6\,|f_n|}{1 - 2\,|f_n|} \tag{10.1}$$

on the unit circle $\mathbb{T}$.

Proof. If $|\varphi_n|^2\,\sigma' - 1 < 0$ then (10.1) follows from (6.28). It follows from (2.12) that

$$|\varphi_n|^2\,\sigma' - 1 = 2\,\frac{\operatorname{Re}(\zeta b_n f_n) - |f_n|^2}{|1-\zeta b_n f_n|^2}, \tag{10.2}$$

which implies that $|\varphi_n|^2\,\sigma' - 1 \geqslant 0$ if and only if $\operatorname{Re}(\zeta b_n f_n) \geqslant |f_n|^2$. Now, let $|\varphi_n|^2\,\sigma' - 1 \geqslant 0$ at ζ, $\zeta \in \mathbb{T}$. By (6.4) we have

$$\frac{|\varphi_n|^2\,\sigma' - 1}{|\varphi_n|^2\,\sigma' + 1} = \operatorname{Re}(\zeta b_n f_n) - |f_n|^2 + \frac{|\varphi_n|^2\,\sigma' - 1}{|\varphi_n|^2\,\sigma' + 1}\operatorname{Re}(\zeta b_n f_n) \tag{10.3}$$

Notice that the fraction in the left-hand side of (10.3) is nonnegative and is bounded by 1. Since $|f_n|^2 \leqslant \operatorname{Re}(\zeta b_n f_n) \leqslant |f_n|$, we obtain from (10.3) that

$$0 \leqslant \frac{|\varphi_n|^2\,\sigma' - 1}{|\varphi_n|^2\,\sigma' + 1} \leqslant 2\,|f_n|,$$

which obviously yields

$$|\varphi_n|^2\,\sigma' - 1 \leqslant \frac{4\,|f_n|}{1 - 2\,|f_n|} \tag{10.4}$$

and therefore (10.1) holds. ▮

Proof of Theorem 11. By (10.1) we obtain

$$\|\,|\varphi_n|^2\,\sigma' - 1\|_\infty = O\left(\frac{1}{n^\alpha}\right),$$

which implies that $\inf_{\mathbb{T}} \sigma' > 0$. It follows that

$$\|\,|\varphi_n|^2 - 1/\sigma'\|_\infty = O\left(\frac{1}{n^\alpha}\right).$$

Notice that $|\varphi_n|^2 = \varphi_n \cdot \bar{\varphi}_n$ is a trigonometric polynomial of order n. Now the result follows by the Bernstein–Zygmund theorem [56, Chap. III, Theorem 13.20]. ▮

Recall that in Section 1, to any probability measure σ on $\mathbb{T}$, we related the family of probability measures σ_λ, $\lambda \in \mathbb{T}$, with Geronimus parameters $(\lambda a_n)_{n \geqslant 0}$. It is clear from (1.3) and from Geronimus' theorem that λf is the Schur function of σ_λ.

Let $\varphi_n(z, \lambda)$, $\psi_n(z, \lambda)$ be the orthogonal polynomials in $L^2(d\sigma_\lambda)$ and in $L^2(d\sigma_{-\lambda})$, respectively. Analyzing the continued fraction (1.16) for F_{σ_λ}, we conclude that only its first term depends on λ.

Now, we apply the notations of Section 3 to the continued fraction (1.6); see (3.4). Let $(s_n)_{n\geqslant 0}$ be the sequence of Möbius transforms for the continued fraction of F_σ, whereas $(s_n^*)_{n\geqslant 0}$ denotes the similar sequence for F_{σ_λ}. We have

$$s_0 = s_0^*, \qquad s_n = s_n^*(n \geqslant 2),$$
$$s_1(w) = \frac{2a_0 z}{1 - a_0 z + w}, \qquad s_1^*(w) = \frac{2\lambda a_0 z}{1 - \lambda a_0 z + w}, \tag{10.5}$$

which by (3.4) implies

$$S_n^*(0) = s_0^* \circ s_1^* \circ s_1^{-1} \circ s_0^{-1} \circ S_n(0). \tag{10.6}$$

Easy algebraic computations show that

$$s_0^* \circ s_1^* \circ s_1^{-1} \circ s_0^{-1}(w) = \frac{(w+1) + \lambda(w-1)}{(w+1) - \lambda(w-1)}. \tag{10.7}$$

We already mentioned in Section 1 that $(\Psi_n^*)_{n\geqslant 0}$ is the sequence of numerators and $(\Phi_n^*)_{n\geqslant 0}$ the sequence of denominators of the continued fraction (1.16). By (10.6–10.7) we obtain

$$\begin{aligned} \varphi_n(z, \lambda) &= \frac{1+\bar\lambda}{2}\, \varphi_n(z) + \frac{1-\bar\lambda}{2}\, \psi_n(z), \\ \varphi_n^*(z, \lambda) &= \frac{1+\lambda}{2}\, \varphi_n^*(z) + \frac{1-\lambda}{2}\, \psi_n^*(z); \end{aligned} \tag{10.8}$$

see [13, Theorem 7.1, (7.4)].

Lemma 10.2. *For every z, $z \in \mathbb{T}$, the map*

$$\lambda \mapsto \lambda b_n(z, \lambda)$$

is a homeomorphism of the unit circle.

Proof. By (10.8) and by (5.5) we obtain

$$\begin{aligned} \lambda b_n(z, \lambda) &= \lambda\, \frac{\varphi_n(z, \lambda)}{\varphi_n^*(z, \lambda)} = \frac{(1+\lambda)\, \varphi_n - (1-\lambda)\, \psi_n}{(1+\lambda)\, \varphi_n^* + (1-\lambda)\, \psi_n^*} \\ &= \frac{(\varphi_n - \psi_n) + \lambda(\varphi_n + \psi_n)}{(\varphi_n^* + \psi_n^*) + \lambda(\varphi_n^* - \psi_n^*)} = \frac{-A_{n-1}^* + \lambda z B_{n-1}^*}{B_{n-1} - \lambda z A_{n-1}} \\ &= \frac{B_{n-1}^*}{B_{n-1}} \cdot \frac{\lambda z - (A_{n-1}^*/B_{n-1}^*)}{1 - \lambda z (A_{n-1}/B_{n-1})}. \end{aligned} \tag{10.9}$$

Observing that for $z \in \mathbb{T}$, $A^*_{n-1}/B^*_{n-1} = \overline{A}_{n-1}/\overline{B}_{n-1}$, and that by (4.6) this complex number lies in $\mathbb{D}$, we obtain from (10.9) that $\lambda \mapsto b_n(z, \lambda)$ is a composition of Möbius transforms of $\mathbb{T}$. ∎

Corollary 10.3. *Let σ be a probability measure on $\mathbb{T}$. Then*

$$|f_n| \leqslant \sup_{\lambda \in \mathbb{T}} \big|\, |\varphi_n(\zeta, \lambda)|^2 \sigma'_\lambda - 1 \big| \tag{10.10}$$

on the unit circle $\mathbb{T}$.

Proof. By Lemma 10.2, given $\zeta \in \mathbb{T}$ there exists $\lambda \in \mathbb{T}$ such that

$$\zeta b_n(\zeta, \lambda) \cdot \lambda \cdot f_n(\zeta) = \operatorname{Re}[\zeta b_n(\zeta, \lambda)\, \lambda f_n] = -|f_n|. \tag{10.11}$$

Using (10.2), we obtain that

$$\sup_{\lambda \in \mathbb{T}} \big|\, |\varphi_n(\zeta, \lambda)|^2 \sigma'_\lambda - 1 \big| \geqslant 2 \frac{|f_n| + |f_n|^2}{(1 + |f_n|)^2} = \frac{2\,|f_n|}{1 + |f_n|} \geqslant |f_n|, \tag{10.12}$$

as stated. ∎

Proof of Theorem 12. Since σ is absolutely continuous and $(\sigma')^{-1} \in \Lambda_\alpha$, its harmonic conjugate function

$$-\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{d\sigma(t)}{2tg\,\dfrac{(t-x)}{2}},$$

see [56, Chap. VII, (1.8)], [56, Chap. III, Theorem 13.29] is in Λ_α. It follows that $F_\sigma \in \Lambda_\alpha$. Since

$$zf = \frac{F_\sigma - 1}{F_\sigma + 1}$$

and obviously $(F_\sigma + 1)^{-1} \in \Lambda_\alpha$, we conclude that $f \in \Lambda_\alpha$. Moreover, $\|f\|_\infty < 1$ since $\inf \sigma' > 0$. Observing that λf is the Schur function of σ_λ, we obtain that

$$\lambda \to (\sigma'_\lambda)^{-1}$$

is a homeomorphism of $\mathbb{T}$ into $\Lambda_\alpha \setminus \{\mathbb{0}\}$.

By Szegő's theorem

$$\varphi_n(z, \lambda) = \frac{z^n}{D(\sigma_\lambda, z)} + O\left(\frac{\log n}{n^\alpha}\right), \qquad n \to +\infty$$

uniformly in z and λ; z, $\lambda \in \mathbb{T}$ [21]. Notice that the proof given in [21, Sect. 3.5] extends to $\alpha > 1$ by Bernstein's theorem on best polynomial approximation. It follows that

$$\sup_{\lambda \in \mathbb{T}} \left| |\varphi_n(\zeta, \lambda)|^2 \sigma'_\lambda - 1 \right| = O\left(\frac{\log n}{n^\alpha}\right)$$

uniformly in ζ, $\zeta \in \mathbb{T}$, which implies (2.36) by Corollary 10.3. ∎

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* Added in proofs.